A NOTE ON THE MUTUAL-VISIBILITY COLORING OF HYPERCUBES

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ABSTRACT. A subset M of vertices in a graph G is a mutual-visibility set if for any two vertices $u, v \in M$ there exists a shortest u-v path in G that contains no elements of M as internal vertices. Let $\chi_\mu(G)$ be the least number of colors needed to color the vertices of G, so that each color class is a mutual-visibility set. Let $n \in \mathbb{N}$ and Q_n be an n-dimensional hypercube. It has been shown that the maximum size of a mutual-visibility set in Q_n is at least $\Omega(2^n)$. Klavžar, Kuziak, Valenzuela-Tripodoro, and Yero further asked whether it is true that $\chi_\mu(Q_n) = O(1)$. In this note we answer their question in the negative by showing that

$$\omega(1) = \chi_\mu(Q_n) = O(\log \log n).$$

1. Introduction

Let G be a simple graph. A subset $M \subseteq V(G)$ of vertices is called a *mutual-visibility set* if any two vertices $u, v \in M$ can "see" each other, that is, there exists a shortest u-v path in G that contains no elements of M as internal vertices. As in all other extremal problems, we are interested in the largest size of a mutual-visibility set in a given graph G, denoted as $\mu(G)$. The systematic investigation was pioneered by Di Stefano [10] and has garnered extensive attention and subsequent research [1,3–9,13,14, etc.] in recent years. As the mutual-visibility problem was initially motivated by establishing efficient and confidential communication in networks, the research on $\mu(G)$ mainly focuses on sparse and highly connected graphs, such as product graphs and hypercube-like graphs. For $n \in \mathbb{N}$, the *n-dimensional hypercube* Q_n is a graph on the vertex set $2^{[n]}$, such that two vertices $A, B \in 2^{[n]}$ form an edge in Q_n if and only if their *symmetric difference* $A \Delta B := (A \backslash B) \cup (B \backslash A)$ has size 1. It is known that Q_n contains a large mutual-visibility set.

Theorem 1 ([1, Theorem 1.2])**.** *For every $n \in \mathbb{N}$, we have $\mu(Q_n) > 0.186 \cdot 2^n$.*

Recently, Klavžar, Kuziak, Valenzuela-Tripodoro, and Yero [12] introduced the coloring version of the mutual-visibility problem. Given a coloring of the vertices of G, we say that G is *properly colored* if each color class is a mutual-visibility set in G. The function considered in their paper is $\chi_\mu(G)$, which is the least number of colors needed to properly color $V(G)$. Equivalently, $\chi_\mu(G)$ is the smallest integer such that $V(G)$ can be partitioned into $\chi_\mu(G)$ mutual-visibility sets. It is easy to see that $\chi_\mu(G) \geq |V(G)|/\mu(G)$. Naturally, one might ask whether $\chi_\mu(G) = O\left(|V(G)|/\mu(G)\right)$ could be true in general. Given that $\mu(Q_n) = \Omega(Q_n)$, Klavžar, Kuziak, Valenzuela-Tripodoro, and Yero [12] raised the following question:

Is there an absolute constant $C > 0$, such that $\chi_\mu(Q_n) \leq C$ holds for all $n \in \mathbb{N}$?

We answer their question in the negative with the following result. In particular, this shows that $\chi_\mu(G)$ can be arbitrarily far from the trivial lower bound $|V(G)|/\mu(G)$.

Theorem 2. $\omega(1) = \chi_\mu(Q_n) = O(\log \log n)$.

2. Proof of Theorem 2: lower bound

Proof of the lower bound. Fix any positive integer q. It suffices to show that there exists $n_0 > 0$, such that $\chi_\mu(Q_n) > q$ holds for all $n \geq n_0$.

Let $n \geq n_0$ with some $n_0 > 0$ to be determined later and fix an arbitrary q-coloring of $V(Q_n)$. We shall argue that Q_n is not properly colored, i.e., some color class is not a mutual-visibility set in Q_n. To do this, we first prove the following claim. Let $n \geq k \in \mathbb{N}$. The *kth layer* of a subgraph $Q \subseteq Q_n$, denoted by $\mathcal{L}_k(Q)$, is defined as $V(Q) \cap \binom{[n]}{k}$.

Claim 3. *Let $n \geq n' \geq 2$ and $Q \subseteq Q_n$ be a copy of $Q_{n'}$. If $M \subseteq V(Q_n)$ contains three layers of Q, then M is not a mutual-visibility set in Q_n.*

Proof of Claim 3. Let $\mathcal{L}_i$, $\mathcal{L}_j$ and $\mathcal{L}_k$ be the three layers of Q contained in M, where $i < j < k$. Take two vertices $A \in \mathcal{L}_i$ and $B \in \mathcal{L}_k$ with $A \subseteq B$. Observe that every shortest A-B path goes through some vertex $C \in V(Q_n)$ satisfying $|C| = j$ and $A \subseteq C \subseteq B$. As Q is a copy of $Q_{n'}$, all such vertices C are contained in $\mathcal{L}_j$. Namely, every shortest A-B path must go through the layer $\mathcal{L}_j$, so M is not a mutual-visibility set. □

To show that Q_n is not properly colored, by Claim 3 it suffices to find a copy of lower dimensional hypercube, which has three layers receiving the same color. Our argument here is a generalization of that in [2, Lemma 2]. Given integers $s \geq k \geq 2$, the *q-color hypergraph Ramsey number* $r_k(s; q)$ is defined as the smallest integer r such that any q-coloring of $\binom{[r]}{k}$ contains a monochromatic copy of $\binom{[s]}{k}$. Since q is fixed in the beginning, for simplicity we write $r_k(s) = r_k(s; q)$. Now let

$$n_0 = q \cdot (r_2 \circ r_3 \circ \cdots \circ r_{2q}(2q)). \tag{2.1}$$

Recall that $V(Q_n) = 2^{[n]}$ and $n \geq n_0$. By the pigeonhole principle there exists $X_1 \subseteq [n]$ with

$$|X_1| \geq n/q \geq r_2 \circ r_3 \circ \cdots \circ r_{2q}(2q), \tag{2.2}$$

such that $\binom{X_1}{1}$ is monochromatic. Then, due to the size of X_1, there exists $X_2 \subseteq X_1$ with

$$|X_2| \geq r_3 \circ \cdots \circ r_{2q}(2q), \tag{2.3}$$

such that $\binom{X_2}{2}$ is monochromatic. By repeating this argument we obtain a sequence of sets

$$X_{2q} \subseteq X_{2q-1} \subseteq \cdots \subseteq X_2 \subseteq X_1 \subseteq [n], \tag{2.4}$$

such that $|X_{2q}| = 2q$ and $\binom{X_k}{k}$ is monochromatic for each $k \in [2q]$. Let Q be the subgraph of Q_n induced by $2^{X_{2q}}$, in particular, Q is a copy of Q_{2q}. We have that every layer of Q is monochromatic. Since Q contains $2q + 1$ layers and there are q colors, by the pigeonhole principle at least three layers of Q receive the same color. This completes the proof of the lower bound in Theorem 2. □

Remark 4. *Although the proof above shows that $\chi_\mu(Q_n)$ grows with n, the dependence of the lower bound on n is quite unsatisfactory. In fact, it is even difficult to express the lower bound on $\chi_\mu(Q_n)$ in terms of n, since for $\chi_\mu(Q_n) > q$ we require n to be at least a composition of q-color hypergraph Ramsey numbers (see* (2.1)*), which is a tower function of q with height roughly $\Theta(q^2)$ (for references on bounds on hypergraph Ramsey numbers, see, e.g., [15]). It would be interesting to actually obtain an expressible lower bound.*

3. Proof of Theorem 2: upper bound

Our proof of the upper bound consists two steps. First, we will reduce the problem from coloring the entire vertex set $V(Q_n)$ to coloring a single layer of Q_n in a certain way. Then, we show that a random coloring of the layer is potentially a good coloring. To achieve the first step, we need the following lemma from [1].

Lemma 5 ([1, Lemma 2.2])**.** *Let $n, g \in \mathbb{N}$, $g \geq 3$. For each $k \in [0, n] := \{0, 1, \ldots, n\}$, let $\mathcal{F}_k \subseteq \binom{[n]}{k}$ be such that*

$$\forall\, A, B \in 2^{[n]} \text{ with } |A| + g \leq k \leq |B| - g,\ \exists\, T \in \binom{[n]}{k} \setminus \mathcal{F}_k :\ A \cap B \subseteq T \subseteq A \cup B. \tag{3.1}$$

Let $\lambda \in [g]$ and $I_\lambda = \{k \in [0, n] : k \equiv \lambda \pmod g\}$. Then $M := \bigcup_{k \in I_\lambda} \mathcal{F}_k$ is a mutual-visibility set in Q_n.

From Lemma 5 we derive the key lemma of our proof.

Lemma 6. *Let $n, g \in \mathbb{N}$, $g \geq 3$. Suppose for every $k \in [0, n]$ we can color $\binom{[n]}{k}$ with q colors so that each color class $\mathcal{F}_k^i$ with $i \in [q]$ satisfies the following*

$$\forall A \in \binom{[n]}{k-g} \text{ and } B \in \binom{[n]}{k+g} \text{ with } A \subseteq B, \exists T \in \binom{[n]}{k} \backslash \mathcal{F}_k : A \subseteq T \subseteq B. \tag{3.2}$$

Then we have $\chi_\mu(Q_n) \leq gq$.

Proof of Lemma 6. First, we show that the property (3.2) is equivalent to the property (3.1). It is obvious that (3.1) implies (3.2). It suffices to show the other direction. Fix any $A, B \in 2^{[n]}$ with $|A| + g \leq k \leq |B| - g$. Since $|A \cap B| \leqslant |A| \leqslant k - g$ and $|A \cup B| \geqslant |B| \geqslant k + g$, there are some $A' \in \binom{[n]}{k-g}$ and $B' \in \binom{[n]}{k+g}$ with $A \cap B \subseteq A' \subseteq B' \subseteq A \cup B$. Then by (3.2) there exists $T \in \binom{[n]}{k} \backslash \mathcal{F}_k$ such that $A' \subseteq T \subseteq B'$, which in particular implies that $A \cap B \subseteq T \subseteq A \cup B$.

Now for each $k \in [0, n]$, we color the layer $\binom{[n]}{k}$ with q colors such that every color class satisfies the property (3.2) and thus the property (3.1). Let $\mathcal{F}_k^i \subseteq \binom{[n]}{k}$ denote the ith color class in the kth layer. It holds that

$$V(Q_n) = \bigcup_{k=0}^{n} \binom{[n]}{k} = \bigcup_{k=0}^{n} \left(\bigcup_{i=1}^{q} \mathcal{F}_k^i \right) = \bigcup_{i \in [q], \lambda \in [g]} \left(\bigcup_{k \in I_\lambda} \mathcal{F}_k^i \right), \tag{3.3}$$

where $I_\lambda = \{k \in [0, n] : k \equiv \lambda \pmod g\}$. By Lemma 5 the set $\bigcup_{k \in I_\lambda} \mathcal{F}_k^i$ is a mutual-visibility set in Q_n. So $V(Q_n)$ can be partitioned into qd mutual-visibility sets, namely, $\chi_\mu(Q_n) \leq gq$. □

Now we are ready to prove the upper bound in Theorem 2.

Proof of the upper bound. Assume that n is sufficiently large and all logarithms are in base 2. By Lemma 6, to prove the stated upper bound it suffices to show that for $g = \lfloor \log \log n \rfloor \geq 3$ we can color every layer $\binom{[n]}{k}$ with at most 2 colors so that each color class satisfies (3.2).

For $k \in [0, n]$ with $k < g$ or $k > n - g$, the whole layer $\binom{[n]}{k}$ satisfies (3.2), because there exists no such pair (A, B) with $A \in \binom{[n]}{k-g}$ and $B \in \binom{[n]}{k+g}$. Hence, the layer $\binom{[n]}{k}$ can be colored with only one color.

For $k \in [0, n]$ with $g \leq k \leq n - g$, we color the layer $\binom{[n]}{k}$ with 2 colors uniformly at random. Let

$$\mathcal{J} := \left\{ (A, B) : A \in \binom{[n]}{k-g} \text{ and } B \in \binom{[n]}{k+g} \text{ with } A \subseteq B \right\}. \tag{3.4}$$

For each $(A, B) \in \mathcal{J}$, we define the event $\mathcal{E}_{(A,B)}$ that all $T \in \binom{[n]}{k}$ with $A \subseteq T \subseteq B$ are colored with the same color. It is not hard to see that

$$\mathbb{P}\left(\mathcal{E}_{(A,B)}\right) \leq 2^{1 - \binom{2g}{g}} =: p. \tag{3.5}$$

Observe that a given event $\mathcal{E}_{(A,B)}$ is mutually independent of all the other events except for those $\mathcal{E}_{(A',B')}$, where there exists $T \in \binom{[n]}{k}$ with $A \subseteq T \subseteq B$ and $A' \subseteq T \subseteq B'$. We shall count the number of such events $\mathcal{E}_{(A',B')}$, denoted by d. First, note that the number of $T \in \binom{[n]}{k}$ with $A \subseteq T \subseteq B$ is $\binom{2g}{g}$. Moreover, for every $T \in \binom{[n]}{k}$, there are $\binom{k}{g}\binom{n-k}{g}$ pairs $(A', B') \in \mathcal{J}$ such that $A' \subseteq T \subseteq B'$. So we have

$$d \leq \binom{2g}{g}\binom{k}{g}\binom{n-k}{g} \leq \binom{2g}{g}\binom{n}{2g} < \left(\frac{en}{g}\right)^{2g}. \tag{3.6}$$

Now since

$$ep(d+1) \le 2^{1-\binom{2g}{g}} n^{2g} \le 2^{-2^{2\log\log n}/\sqrt{100\log\log n}} n^{2\log\log n} \le 2^{-(\log n)^2/\sqrt{100\log\log n}+2\log n\log\log n} \le 1, \tag{3.7}$$

by the celebrated Lovász Local Lemma [11] (see also [16, Theorem 1.5]), there is a positive probability that none of the events $\mathcal{E}_{(A,B)}$ with $(A,B) \in \mathcal{J}$ occurs. Namely, there is a coloring of $\binom{[n]}{k}$ with 2 colors such that both color classes satisfy (3.2), from which the upper bound follows. □

References

[1] Maria Axenovich and Dingyuan Liu. "Visibility in Hypercubes." arXiv preprint arXiv:2402.04791 (2024).
[2] Maria Axenovich and Stefan Walzer. "Boolean lattices: Ramsey properties and embeddings." *Order* **34** (2017): 287–298.
[3] Gülnaz Boruzanli Ekinci and Csilla Bujtás. "Mutual-visibility problems in Kneser and Johnson graphs." arXiv preprint arXiv:2403.15645 (2024).
[4] Boštjan Brešar and Ismael G. Yero. "Lower (total) mutual-visibility number in graphs." *Applied Mathematics and Computation* **465** (2024).
[5] Serafino Cicerone, Alessia Di Fonso, Gabriele Di Stefano, Alfredo Navarra, and Francesco Piselli. "Mutual visibility in hypercube-like graphs." *International Colloquium on Structural Information and Communication Complexity* (2024): 192–207.
[6] Serafino Cicerone, Gabriele Di Stefano, Lara Drožđek, Jaka Hedžet, Sandi Klavžar, and Ismael G. Yero. "Variety of mutual-visibility problems in graphs." *Theoretical Computer Science* **974** (2023).
[7] Serafino Cicerone, Gabriele Di Stefano, and Sandi Klavžar. "On the mutual visibility in Cartesian products and triangle-free graphs." *Applied Mathematics and Computation* **438** (2023).
[8] Serafino Cicerone, Gabriele Di Stefano, Sandi Klavžar, and Ismael G. Yero. "Mutual-visibility in strong products of graphs via total mutual-visibility." *Discrete Applied Mathematics* **358** (2024): 136–146.
[9] Serafino Cicerone, Gabriele Di Stefano, Sandi Klavžar, and Ismael G. Yero. "Mutual-visibility problems on graphs of diameter two." *European Journal of Combinatorics* **120** (2024).
[10] Gabriele Di Stefano. "Mutual visibility in graphs." *Applied Mathematics and Computation* **419** (2022).
[11] Paul Erdős and László Lovász. "Problems and results on 3-chromatic hypergraphs and some related questions." *Colloquia mathematica Societatis János Bolyai* **10** (1973): 609–627.
[12] Sandi Klavžar, Dorota Kuziak, Juan Carlos Valenzuela Tripodoro, and Ismael G. Yero. "Coloring the vertices of a graph with mutual-visibility property." arXiv preprint arXiv:2408.03132 (2024).
[13] Danilo Korže and Aleksander Vesel. "Mutual-visibility sets in Cartesian products of paths and cycles." *Results in Mathematics* **79** (2024).
[14] Danilo Korže and Aleksander Vesel. "Variety of mutual-visibility problems in hypercubes." arXiv preprint arXiv:2405.05650 (2024).
[15] Dhruv Mubayi and Andrew Suk. "A Survey of Hypergraph Ramsey Problems." *Discrete Mathematics and Applications* **165** (2020): 405–428.
[16] Joel Spencer. "Asymptotic lower bounds for Ramsey functions." *Discrete Mathematics* **20** (1977): 69–76.

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