Long path and cycle decompositions of even hypercubes

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Abstract

We consider edge decompositions of the n-dimensional hypercube Q_n into isomorphic copies of a given graph H. While a number of results are known about decomposing Q_n into graphs from various classes, the simplest cases of paths and cycles of a given length are far from being understood. A conjecture of Erde asserts that if n is even, $\ell < 2^n$ and ℓ divides the number of edges of Q_n, then the path of length ℓ decomposes Q_n. Tapadia et al. proved that any path of length $2^m n$, where $2^m < n$, that satisfies these conditions decomposes Q_n. Here, we make progress toward resolving Erde's conjecture by showing that Q_n can be decomposed into cycles of lengths up to $2^{n+1}/n$. As a consequence, we also obtain results about decomposing Q_n into paths of lengths up to $2^n/n$.

1 Introduction

The n-dimensional hypercube Q_n is the graph with $V(Q_n) = \{0,1\}^n$ and edges between pairs of vertices that differ in exactly one coordinate. Given a graph H, we say that H decomposes Q_n if Q_n is a pairwise edge-disjoint union of isomorphic copies of H. For any fixed graph H which is a subgraph of some hypercube, Offner [17] showed that H almost decomposes any Q_n for sufficiently large n. More precisely, a subgraph of Q_n with all but at most $o(|E(Q_n)|)$ edges of Q_n is a pairwise edge-disjoint union of isomorphic copies of H. Aubert and Schneider [3] proved that when n is even Q_n has a decomposition into Hamiltonian cycles. Here, we focus on decompositions of hypercubes into cycles and paths of given length.

If n is odd then each vertex of Q_n has odd degree and hence must be an endpoint of some path in a path decomposition. This implies that there are at least 2^{n-1} paths in such a decomposition and the length of each such path is at most $|E(Q_n)|/2^{n-1} = n2^{n-1}/2^{n-1} = n$. In fact, Anick and Ramras [2] as well as Erde [8] proved that for odd n, Q_n can be decomposed by any path of length at most n and dividing the number of edges in Q_n. While for odd

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n, we can only hope for path decompositions into short paths, when n is even, Erde [8] formulated the following strong conjecture that implies that there are path decompositions of hypercubes into long paths.

Conjecture 1 (Erde [8])**.** If n is even, $\ell < 2^n$, and ℓ divides the number of edges of Q_n, then the path of length ℓ decomposes Q_n.

Here, we prove that there are cycle decompositions of hypercubes of even dimension into long cycles, from which it follows that there are decompositions of such hypercubes into long paths. The best known result is by Tapadia et al. [22] (see also Horak et al. [13]) which gives cycle decompositions of Q_n into cycles of length at most n^2.

Theorem (Tapadia et al. [22])**.** *Let n and m be integers where n is positive and even and m is nonnegative, such that $2^m \leq n$. Then a cycle of length $2^m n$ decomposes Q_n.*

Note that the number of edges in Q_n is $n2^{n-1}$. So for even n, if there is a cycle decomposition of Q_n into cycles of length ℓ, then $\ell = y2^m$, where y is an odd divisor of n. We will show that for any odd divisor y of n, there is a cycle decomposition of Q_n into cycles of length $y2^m$, where m can take a range of values.

Theorem 1. *Let $n = xy2^\alpha$, where $x, y \geq 1$ are odd, and $\alpha \geq 1$. Suppose y has binary representation $y = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_j}$, where $i_1 > i_2 > \cdots > i_j = 0$. Then for any q, $0 \leq q \leq n - i_1 - 2xj$, Q_n has an edge decomposition into $x2^{i_1+\alpha+j-2+q}$ cycles of length $y2^{n-i_1-j-q+1}$.*

As an example, consider Q_{30}, where $\alpha = 1$. Letting $x = 3$ and $y = 5 = 2^2 + 2^0$ gives $i_1 = 2$ and $j = 2$, so $n - i_1 - 2xj = 16$. Thus we get decompositions into $x2^{i_1+\alpha+j-2+q} = 3 \cdot 2^{3+q}$ cycles, for $0 \leq q \leq 16$. Since Q_{30} has $30 \cdot 2^{29}$ edges, the cycle lengths of these decompositions are $\{30 \cdot 2^{29}/3 \cdot 2^n \ : \ 3 \leq n \leq 19\} = \{5 \cdot 2^m \ : \ 11 \leq m \leq 27\}$. See Table 1 in the appendix for further numerical examples.

The rough idea of the proof is as follows. We represent Q_n as a Cartesian product of smaller hypercubes. By induction, using the result of Aubert and Schneider [3] as a base case, we decompose each of the smaller hypercubes into cycles. We consider the products of these cycles from different copies of the smaller hypercubes. The Cartesian product of two cycles forms a toroidal grid (which we refer to simply as a torus), and in Section 3.2 we show how to decompose a torus into several cycles of the same length using what we call a “wiggle” decomposition. In the actual proof we treat special subdivided tori in a similar fashion, where we carefully control the subdivisions so that the resulting cycles are all the same length.

Note that by splitting each cycle in a cycle decomposition of Q_n into paths of equal length we obtain path decompositions of Q_n. In particular, we see that there are path decompositions of Q_n into any path of length at most $2^n/n$ where the length divides $n2^{n-1}$. By taking $x = 1$ and $q = 0$ in Theorem 1, we obtain the following.

Corollary. *Let n be even. Then there is a decomposition of Q_n into cycles of length ℓ, where $\ell \geq 2^{n+1}/n$ and ℓ is divisible by the largest odd divisor of n. In particular, Q_n has a* *decomposition into paths of length ℓ', for all ℓ' dividing the number of edges in Q_n such that $\ell' \leq 2^n/n$.*

The paper is structured as follows. We give more background and historical information on hypercube decompositions in Section 2. In Section 3, we introduce the wiggle decomposition for decomposing tori and subdivided tori into cycles. We also introduce stronger notions of splittable and DR-splittable decompositions, and show how to produce these type of cycle decompositions of tori and subdivided tori. In Section 4 we state several general decomposition results on Cartesian products, and show how given cycle decompositions of graphs G and G' we can produce a cycle decomposition of their Cartesian product with all cycles the same length. Finally, in Section 5 we prove the main theorem and in Section 6 we offer some conclusions.

2 Background

For a graph $G = (V, E)$, we say that a graph H *divides* the graph G if the greatest common divisor of the degrees of H divides the greatest common divisor of the degrees of G and $|E(H)|$ divides $|E(G)|$. We call a subgraph of G isomorphic to H a *copy* of H in G. We use K_n to denote a complete graph on n vertices. A classical theorem of Wilson [23] states that for any graph H, if n is sufficiently large and H divides K_n then H decomposes K_n. This result was generalized for subgraphs G of K_n with sufficiently large minimum degree and graphs H dividing G, see Keevash [14] and Glock et al. [11]. Given Wilson's result on K_n, it was natural to consider the analogous problem with other ground graphs, for example a hypercube.

A graph H is called *cubical* if it is a subgraph of Q_n for some n. It is clear that only graphs which are cubical and divide Q_n can decompose Q_n. However, unlike the above results for dense subgraphs of K_n, these properties are not sufficient for decomposing Q_n, as shown by a counterexample of Bonamy et al. [7].

The initial results involving packings and decompositions of the hypercube are due to Stout [21] and were motivated by processor allocation problems. He introduced both the notion of vertex packing and edge packing of the hypercube and proved an asymptotically optimal result for vertex packing. He showed that for any cubical graph H, there are pairwise vertex disjoint copies of H in Q_n containing all but $o(|V(Q_n)|)$ vertices of Q_n. Answering a question of Offner, Gruslys [12] strengthened Stout's result on vertex packing by proving that if the order of H is a power of 2, then for sufficiently large n, there are pairwise vertex-disjoint copies of H containing all vertices of Q_n. In fact, Gruslys's result holds even for the stronger notion of isometric embeddings.

Stout [21] proved a number of results about edge packing of graphs in Q_n. For example, he showed that if T is a tree with n edges, then T decomposes Q_n, a result independently proved by Fink [9]. Stout conjectured that for any cubical graph H there are pairwise edge-disjoint copies of H in Q_n containing all but $o(|E(Q_n)|)$ edges of Q_n. This conjecture was later proved by Offner [17]. A fan with a root vertex v is a graph which is a union of cycles of the same length that pairwise share only v. A double-fan is the graph obtained by joining the root vertices of two vertex disjoint fans by an edge. In [19], Roy and Kureethara proved several results about decomposing Q_n into fans and double-fans. Horak et al. [13] showed that if H is a cubical graph of size n, each block of which is either a cycle or an edge, then H decomposes Q_n.

A major direction in the decomposition literature concerns Hamiltonian decompositions, that is decompositions into Hamiltonian cycles or Hamiltonian cycles and a perfect matching, see for example a survey of Alspach et al. [1]. Investigations of Hamiltonian decompositions of K_n were carried out as early as the 1800's by Walecki in [16]. His constructions showed that K_n has a Hamiltonian decomposition for all n and a decomposition into Hamiltonian paths for even n. This result was extended by Auerbach and Laskar [4], who showed that complete multipartite graphs with parts of equal size have Hamiltonian decompositions. Ringel [18] proved that Q_n has a Hamiltonian decomposition for all integers n which are powers of 2 and asked whether Q_n has a Hamiltonian decomposition for all even n.

Closely relevant to cycle decompositions of Q_n are Hamiltonian cycle decompositions of the product of cycles. Kotzig [15] proved that the Cartesian product of any two cycles is decomposable into Hamiltonian cycles. This result was extended to products of three cycles by Foregger [10], who in the process gave an alternative proof of Kotzig's result. Finally, Aubert and Schneider [3] extended Foregger's result by proving a general theorem which implies that a product of arbitrarily many cycles has a Hamiltonian decomposition. One consequence of their results is a solution to Ringel's problem of showing that Q_n has a Hamiltonian decomposition when n is even, since Q_n is the Cartesian product of $n/2$ cycles of length 4.

An important open problem for Hamiltonian decompositions is a conjecture of Bermond [6] asserting that the Cartesian product of two graphs with a Hamiltonian decomposition has a Hamiltonian decomposition. This conjecture has been settled under fairly general conditions by Stong [20] but remains open in general. Motivated by problems in parallel computing, Bass and Sudborough [5] considered decompositions of Q_n into k-regular spanning subgraphs.

3 Cycle decompositions of tori and subdivided tori

We begin with some notation which we will need for the notions in this section. For graphs G and H, denote by $G \cup H$ the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. We denote by $G \,\square\, H$ the Cartesian product of G and H, i.e., a graph with vertex set $\{(u,v) : u \in V(G), v \in V(H)\}$ and edge set $\{(u,v)(u',v') : u = u', vv' \in E(H) \text{ or } v = v', uu' \in E(G)\}$. We use the notation (e,v) and (u,e') for an edge $(u,v)(u',v)$, $e = uu'$ and for an edge $(u,v)(u,v')$, $e' = vv'$, respectively. We think of (e,v) as a "vertical" edge and (u,e') as a "horizontal" edge in a usual grid drawing of Cartesian product. For a fixed $e \in E(G)$, we call the set of edges $\{(e,v) : v \in V(H)\}$ an *edge row* or just a *row*. Similarly, for a fixed $e' \in E(H)$, we call the edges $\{(u,e') : u \in V(G)\}$ an *edge column* or just a *column*. Note that in our convention the edges in a row are oriented vertically, and those in a column are oriented horizontally. If $G_1, \ldots, G_k$ are subgraphs of G, we say the set

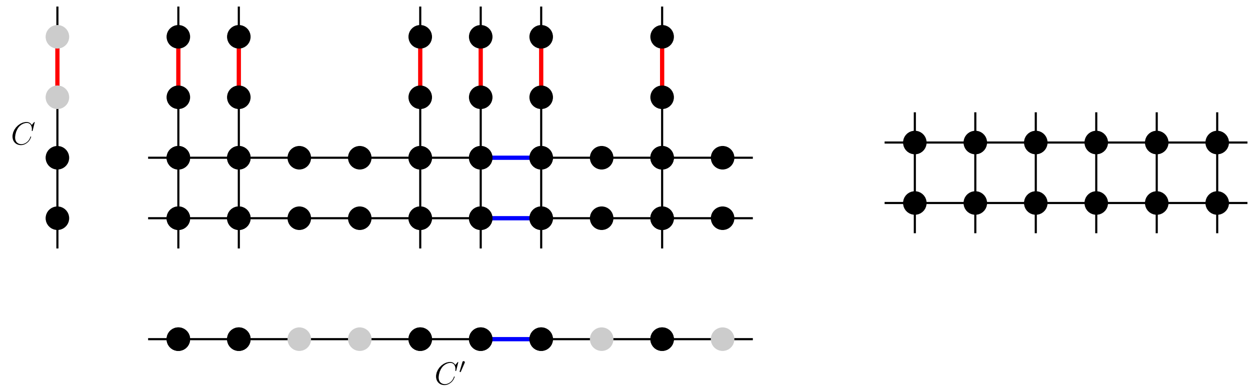

Figure 1: Left: Two cycles $C = (1, 2, 3, 4, 1)$ and $C' = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1)$, and their anchored product $(C, S) \boxplus (C', S')$, where $S = \{1, 2\}$ and $S' = \{1, 2, 5, 6, 7, 9\}$. A given row and column of the anchored product are highlighted in red and blue, respectively, along with the corresponding edge from the original cycle. Right: The underlying torus of $(C, S) \boxplus (C', S')$.

of graphs $\{G_1, \ldots, G_k\}$ forms a *decomposition* of G if $G = G_1 \cup \cdots \cup G_k$ and the subgraphs are pairwise edge-disjoint. We say the decomposition is a *cycle decomposition* if $G_1, \ldots, G_k$ are all cycles. In this paper we are interested in cycle decompositions where all of the cycles have the same length.

3.1 Anchored products of graphs and subdivided tori

Given graphs G and G' with vertex sets $S \subseteq V(G)$, $S' \subseteq V(G')$, we define the *anchored product* $(G, S) \boxplus (G', S')$ of the pairs (G, S) and (G', S') to be the graph with the vertex set

$$\{(u, v) : u \in V(G), v \in V(G'), \text{ and } u \in S \text{ or } v \in S'\}$$

and edge set

$$\{(u, v)(u', v') : uu' \in E(G),\ v = v' \in S'\} \cup \{(u, v)(u', v') : u = u' \in S,\ vv' \in E(G')\},$$

see Figure 1. Note that if $S = V(G)$ and $S' = V(G')$, the anchored product $(G, S) \boxplus (G', S')$ is the same graph as the Cartesian product $G \,\square\, G'$.

We call the Cartesian product of two cycles $C \,\square\, C'$ a *torus*. Given $v \in V(C)$, we call the cycle induced in $C \,\square\, C'$ by $\{v\} \times V(C')$ a *horizontal cycle*, and given $v' \in V(C')$, we call the cycle induced in $C \,\square\, C'$ by $V(C) \times \{v'\}$ a *vertical cycle*. A *subdivided torus* is a graph obtained from a torus by subdividing edges so that all edges in each row are subdivided by the same number of vertices and all edges in each column are subdivided by the same number of vertices. More formally, a graph F is a subdivided torus if for some cycles C and C' and vertex sets $S \subseteq V(C)$ and $S' \subseteq V(C')$, $F = (C, S) \boxplus (C', S')$. Note that a vertex has degree four in a subdivided torus if and only if it is in $S \times S'$, and otherwise it has degree two. We also see that a subdivided torus is a subgraph of a larger torus $C \,\square\, C'$ and a subdivision of a smaller torus obtained by contracting all degree two vertices. We refer to this smaller torus as the *underlying torus* of the subdivided torus. Note that the underlying torus of F is a Cartesian product of two cycles with lengths $|S|$ and $|S'|$, respectively. The set of edges of a row of $C \,\square\, C'$ that are in F is called a *row of a subdivided torus*. The columns are defined similarly. Figures 1 and 5 show examples of subdivided tori along with their undelying tori. Note that, as in Figure 1, the underlying torus may be a product of a cycle of length 2 with another cycle.

3.2 The k-wiggle decomposition on tori and subdivided tori

Let $k \geq 2$ be an integer. We define a method for decomposing a torus that is product of a cycle C of length divisible by k and a cycle C' of length at least k and congruent to $k \pmod 2$ into k cycles of equal length called the *k-wiggle decomposition*. Let $C = (0, 1, \dots, , n-1, 0)$ be a cycle of length n and $C' = (0, 1, \dots, m-1, 0)$ a cycle of length m, where k is a divisor of n and $m = 2s + k$ for some integer $s \geq 0$. We say that a torus T *allows the k-wiggle decomposition* if it meets these conditions. In the important case $k = 2$, the condition for allowing the k-wiggle decomposition is equivalent to n and m being even. A decomposition of the torus $C \,\square\, C'$ into k cycles $C_1, \dots, C_k$, is called the *k-wiggle decomposition*, if for $\ell = 1, \dots, k$,

$$\begin{aligned}
E(C_\ell) =& \{(i,j)(i+1,j) :\ 0 \leq j \leq m-k-1,\ i \equiv \ell \pmod k\} \\
& \cup \{(i,j)(i+1,j) :\ 0 \leq p \leq k-1,\ i \equiv \ell + p \pmod k,\ j = m-k+p\} \\
& \cup \{(i,j)(i,j+1) :\ i \equiv \ell \pmod k,\ 0 \leq j \leq m-k-1,\ j \text{ odd}\} \\
& \cup \{(i,j)(i,j+1) :\ i \equiv \ell + 1 \pmod k,\ 0 \leq j \leq m-k-1,\ j \text{ even}\} \\
& \cup \{(i,j)(i,j+1) :\ 0 \leq p \leq k-1, j = m-k+p,\ i \equiv \ell + p + 1 \pmod k\}.
\end{aligned}$$

See Figure 2 for examples of the k-wiggle decomposition on Cartesian products of cycles for various k. Note that all cycles in a k-wiggle decomposition on a torus have the same length, and further, the cycles are all vertical translations of each other, i.e. the vertex $(i,j) \in V(C_1)$ if and only if the vertex $(i, j+\ell-1) \in V(C_\ell)$ and the edge $(i,j)(i',j') \in E(C_1)$ if and only if the edge $(i, j+\ell-1)(i', j'+\ell-1) \in E(C_\ell)$.

Consider now a subdivided torus $F = (C, S) \boxplus (C', S')$ such that its underlying torus T allows a k-wiggle decomposition, i.e., $|S|$ is a multiple of k and $|S'|$ is at least k and congruent to k modulo 2. We define a k-wiggle decomposition of F as a decomposition obtained from the k-wiggle decomposition of T by subdividing respective edges. More precisely, if an edge e is in the ith cycle of the decomposition of T, we let all edges of F obtained by subdividing e be in the ith cycle of the decomposition of F. See Figure 5.

The k-wiggle decomposition on a subdivided torus may not produce cycles of all the same length, for example if exactly one vertical edge of C is subdivided. Next we give sufficient

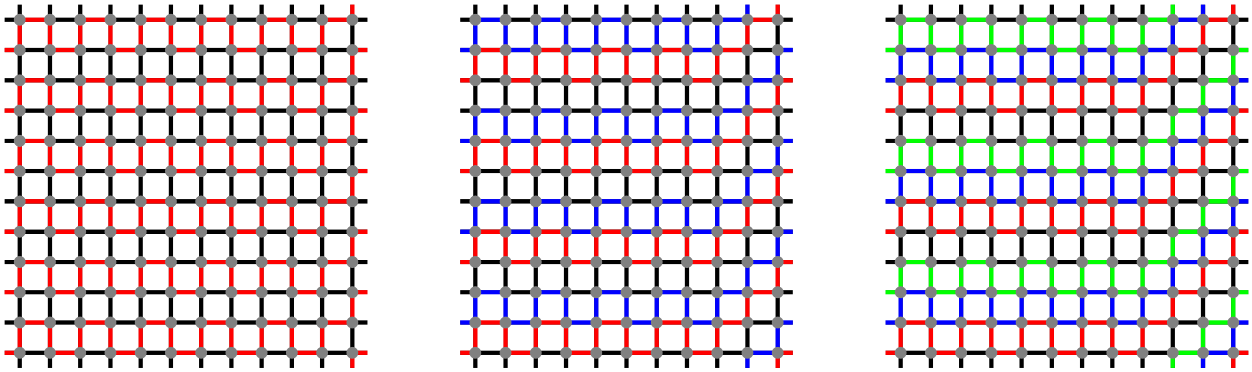

Figure 2: Examples of 2-wiggle (left), 3-wiggle (middle), and 4-wiggle (right) decompositions on the Cartesian product of two cycles.

conditions on the subdivided torus to guarantee the cycles of the k-wiggle decomposition are all the same length. Let C be a cycle, $S \subseteq V(C)$. We say the pair (C, S) is *distance regular* if, when following the cycle in a given direction, every path between consecutive elements of S has the same length.

Proposition 2. *Let C and C' be cycles, $S \subseteq V(C)$, where (C, S) is distance regular, and $S' \subseteq V(C')$. Assume that the underlying torus of $(C, S) \boxplus (C', S')$ allows the k-wiggle decomposition. Then the k-wiggle decomposition on $(C, S) \boxplus (C', S')$ yields k cycles of the same length.*

See Figure 5 for an illustration with $k = 2$. In the figure, $|S| = 4$, $|S'| = 8$, and (C, S) is distance regular as each path between consecutive elements of S has length 2. Each cycle has 52 edges.

Proof. For $1 \leq \ell \leq k$, C_ℓ has $|S|/k$ edges in each column, and thus each cycle has the same number of horizontal edges in the subdivided torus. For $1 \leq \ell \leq k$, C_ℓ has $|S'| - k + 1$ edges in each row whose edges were obtained in a subdivision of the edges from the row of index congruent to $\ell \pmod{k}$ in the underlying torus, and 1 edge in each other row. Since the union of edges in all rows form vertical copies of C and (C, S) is distance regular, all k cycles have the same number of vertical edges. Thus every cycle has the same length. □

The conclusion of Proposition 2 also holds under the weaker assumption that the sum of the lengths of every kth path in $(C, S)$is identical. For example, (C, S) would meet this condition when $k = 3$ if the consecutive path lengths were 1, 1, 2, 1, 3, 2, 3, 1, 1, since the sum of the length of every third path is 5. However we will not need this generality so we use the simpler distance regular condition.

3.3 Splittable decompositions

In this section we define splittable decompositions, and prove some related properties about k-wiggle decompositions of subdivided tori.

A set of graphs $\{G_1, \ldots, G_a\}$ forms a *splittable decomposition* of a graph G if it is a decomposition of G and for $i = 1, \ldots, a$, there are pairwise disjoint sets $S_i \subseteq V(G_i)$ with $|S_1| = |S_2| = \cdots = |S_a| \geq 2$, whose union is $V(G)$. We refer to the sets $S_1, \ldots, S_a$ as *representing sets* of the decomposition.

For $a, m \geq 1$, if the set of graphs $\{G_1, \ldots, G_{am}\}$ is a decomposition of a graph G, we say it forms an *a-splittable decomposition* of G if the set $\{G_1, \ldots, G_{am}\}$ can be partitioned into m pairwise disjoint subsets $\mathcal{F}_1, \ldots, \mathcal{F}_m$, each containing a graphs, such that the graphs in each $\mathcal{F}_i$, $i = 1, \ldots, m$ form a splittable decomposition of a spanning subgraph of G. We call these $\mathcal{F}_i$ the *splitting sets* of the decomposition. An a-splittable decomposition of G is called an *(a, b)-splittable decomposition* if each $\mathcal{F}_i$ can be partitioned into subsets $\mathcal{F}_{i,1}, \ldots, \mathcal{F}_{i,a/b}$, each of cardinality b, where the graphs in $\mathcal{F}_{i,j}$ are pairwise vertex disjoint and span $V(G)$. We call these $\mathcal{F}_{i,j}$ the *splitting subsets* of the decomposition. Note that if all of the graphs in an (a, b)-splittable decomposition have the same number of vertices v, then $b = |V(G)|/v$.

Note that a decomposition $\{G_1, \ldots, G_a\}$ of G is 1-splittable if and only if each graph G_i is a spanning subgraph of G. We call such a decomposition a *spanning decomposition*, and in the case of a cycle decomposition, we call it a *Hamiltonian decomposition*, since every graph in the decomposition is a Hamiltonian cycle. Note that for any a, an $(a, 1)$-splittable decomposition is also a spanning decomposition and an a-splittable decomposition $\{G_1, \ldots, G_a\}$ of G with a graphs is simply a splittable decomposition. We shall use each notion when convenient.

An a-splittable (resp. (a, b)-splittable) cycle decomposition of a graph G is called *a-DR-splittable* (resp. *(a, b)-DR-splittable*) if in addition to the other conditions, for all cycles C in the decomposition, if S is the representing set for C, then (C, S) is distance regular.

Proposition 3. *The decomposition into cycles produced by the k-wiggle decomposition on a torus is k-DR-splittable. If k is even, the decomposition is also $k/2$-DR-splittable.*

Proof. Let $C_1, \ldots, C_k$ be the the cycles in the k-wiggle decomposition of a torus T. For $1 \leq \ell \leq k$ we need to find subsets $S_\ell \subseteq C_\ell$, all of the same cardinality, partitioning $V(T)$ and splitting the cycles into paths of equal length. Let S_1 be the set consisting of every other vertex encountered as C_1 is being traversed in a given direction. For $2 \leq \ell \leq k$, let S_ℓ be the vertical translation of S_1 by $\ell - 1$, i.e., the vertex $(i, j) \in S_1$ if and only if the vertex $(i, j + \ell - 1) \in S_\ell$. Note that every kth vertex in each vertical cycle is part of a given S_ℓ, so these sets partition $V(T)$ and have the same cardinality. Further, since the cycles are all vertical translations of each other, for all ℓ, S_ℓ is the set consisting of every other vertex of $V(C_\ell)$ encountered as C_ℓ is being traversed in a given direction. Thus every path in C_ℓ between consecutive elements of S_ℓ has length 2, and (C_ℓ, S_ℓ) is distance regular, see Figure 3 (left).

Let k be even. To show that the decomposition is $k/2$-DR-splittable, we need to partition the cycles into two splitting sets of $k/2$ cycles each and for each splitting set find splitting subsets of vertices in each cycle of the same cardinality, partitioning $V(T)$ and dividing the cycles into paths of equal length. Let the first splitting set $\mathcal{F}_1$ contain the $k/2$ cycles with odd indices, $\mathcal{F}_1 = \{C_1, C_3, \ldots, C_{k-1}\}$, and the second splitting set $\mathcal{F}_2$ contain the $k/2$ cycles with even indices, $\mathcal{F}_2 = \{C_2, C_4, \ldots, C_k\}$. For each cycle C_ℓ, let $S_\ell = V(C_\ell)$. Then since

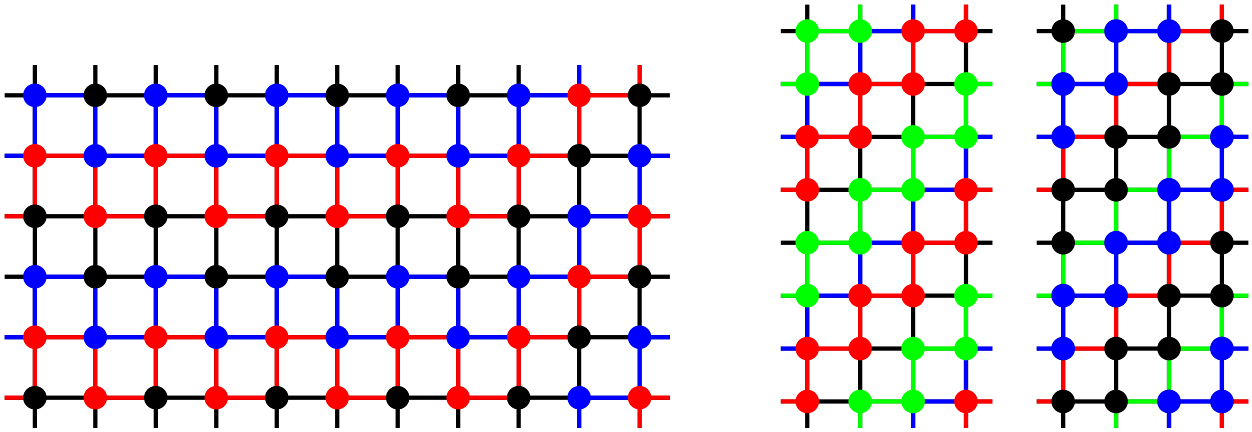

Figure 3: Left: An illustration of the 3-splittable cycle decomposition produced by the 3-wiggle decomposition of a torus. Right: An illustration of the 2-splittable cycle decomposition produced by the 4-wiggle decomposition of the product of an 8-cycle and a 4-cycle. Note that the red and green cycles split the vertex set (left), as do the black and blue cycles (right).

every vertex in the torus is contained in one even-indexed cycle and one odd-indexed cycle, the representative sets in each splitting set partition $V(T)$ and every path in C_ℓ between consecutive elements of S_ℓ has length 1, see Figure 3 (right).

Note that if the decomposition of a torus obtained by the k-wiggle decomposition is a-splittable, then a must be k or $k/2$, as each cycle covers exactly $2/k$ proportion of the vertices in each vertical cycle. Thus at least half of the k cycles are required to cover all the vertices in a given vertical cycle, so at least half of the k cycles are required to cover all the vertices in the torus. □

Proposition 4. *Suppose the torus $C \,\square\, C'$ allows the k-wiggle decomposition and there is a set $S' \subseteq V(C')$ such that (C', S') is distance regular. Then there are sets $S_1, \dots, S_k$, each of the same cardinality, partitioning $V(C) \times S'$ such that for the cycles $C_1, \dots, C_k$ produced by the k-wiggle decomposition on $C \,\square\, C'$, for $1 \le \ell \le k$, (C_ℓ, S_ℓ) is distance regular.*

Proof. Let S_1 be the set consisting of every other vertex of $(V(C) \times S') \cap V(C_1)$ encountered as C_1 is being traversed in a given direction. For $2 \le \ell \le k$, let S_ℓ be the vertical translation of S_1 by $\ell - 1$, i.e., the vertex $(i, j) \in S_1$ if and only if the vertex $(i, j + \ell - 1) \in S_\ell$. Note that every kth vertex in each vertical cycle is part of a given S_ℓ, so these sets partition $V(C) \times S'$ and have the same cardinality, and for all ℓ, S_ℓ is the set consisting of every other vertex of $(V(C) \times S') \cap V(C_\ell)$ encountered as C_ℓ is being traversed in a given direction. Thus every path in C_ℓ between consecutive elements of S_ℓ is twice as long as the corresponding path in the horizontal cycle C' between consecutive elements of S', and (C_ℓ, S_ℓ) is distance regular if and only if (C', S') is. See Figure 4. □

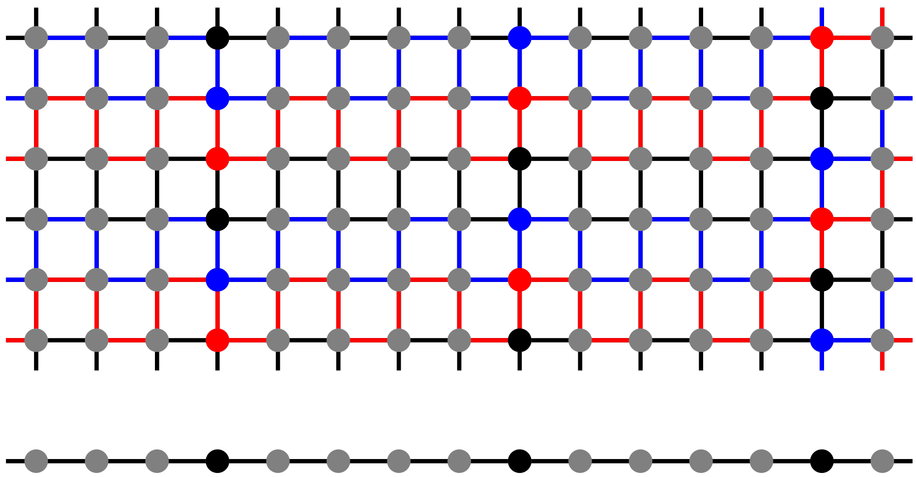

Figure 4: An example for Proposition 4 with $S' = \{4, 9, 14\}$. Note that each path in C' between consecutive elements of S' has length 5, and each corresponding path in the product graph has length 10.

Proposition 5. *Suppose the subdivided torus $(C, S) \boxplus (C', S')$ allows the k-wiggle decomposition, (C, S) is distance regular, and $C_1, \ldots, C_k$ are the cycles produced by the k-wiggle decomposition on $(C, S) \boxplus (C', S')$. Then there are sets $S_1, \ldots, S_k$, each of the same cardinality, partitioning $V(C) \times S'$ such that for $1 \le \ell \le k$, $S_\ell \subseteq V(C_\ell)$.*

Proof. All degree two vertices in the subdivided torus that are in $V(C) \times S'$ lie on only one C_ℓ, and so go in the corresponding S_ℓ. The fact that (C, S) is distance regular and each cycle contains every kth path in each vertical cycle guarantees that there are the same number of each of these in each S_ℓ. It remains to assign the degree four vertices in $V(C) \times S'$, so we ignore the degree two vertices, and consider the underlying torus, with vertex set $S \times S'$. We assign the vertices of the underlying torus to $S_1, \ldots, S_k$ in the alternating pattern of Propositions 3 and 4, so that every other degree 4 vertex on a given cycle is in its representing set. See Figure 5. □

4 Decompositions of Cartesian products of graphs

The main result in this section is Lemma 8, which will be the key tool for inductively generating cycle decompositions on the hypercube. First we need two general statements about decompositions of Cartesian product graphs.

Proposition 6. *Let the graphs $G_1, \ldots, G_a$ form a splittable decomposition of G with representing sets $S_1, \ldots, S_a$ and the graphs $G'_1, \ldots, G'_b$ form a splittable decomposition of G' with*

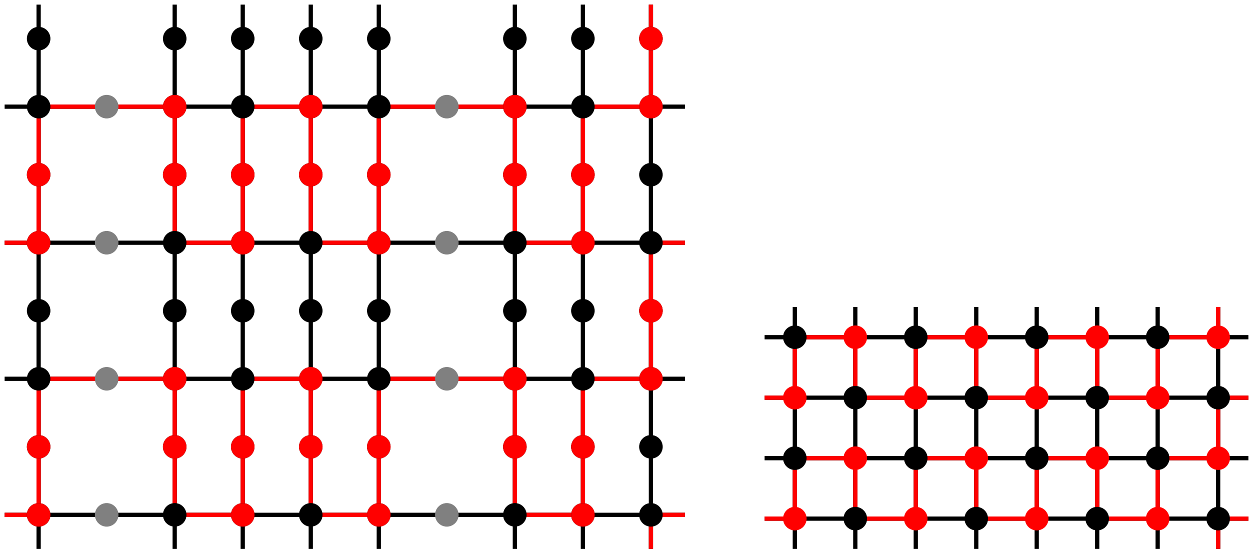

Figure 5: Left: A 2-wiggle decomposition of a subdivided torus. Note that since (C, S) is distance regular (though (C', S') is not) the cycles have the same length. The colors of the vertices correspond to the partiton of $V(C) \times S'$ in Proposition 5. Right: The 2-wiggle decomposition on the underlying torus, where every other vertex on a given cycle is in its representing set.

representing sets $S'_1, \dots, S'_b$. Then

$$G \,\square\, G' = (G_1 \cup \cdots \cup G_a) \,\square\, (G'_1 \cup \cdots \cup G'_b) = \bigcup_{i=1}^{a} \bigcup_{j=1}^{b} (G_i, S_i) \boxplus (G'_j, S'_j),$$

where the union is pairwise disjoint, i.e., a decomposition.

Proof. We shall verify that every edge of $F = (G_1 \cup \cdots \cup G_a) \,\square\, (G'_1 \cup \cdots \cup G'_b)$ is accounted for exactly once in the above union of anchored products. Let $e \in E(F)$, where without loss of generality $e = (u, v)(u', v)$ for $u, u' \in V(G_i)$ and $v \in S'_j$. Then we see that $e \in E((G_i, S_i) \boxplus (G'_j, S'_j))$. Now, consider $e \in E((G_i, S_i) \boxplus (G'_j, S'_j))$, then $e \in E(G_i \,\square\, G'_j) \subseteq E(F)$. Finally, we need to check that no edge of e belongs to two different anchored products $(G_i, S_i) \boxplus (G'_j, S'_j)$ and $(G_q, S_q) \boxplus (G'_p, S'_p)$. Since these products are different, assume without loss of generality that $p \neq j$. Thus $S'_p \cap S'_j = \emptyset$. If $e \in E((G_i, S_i) \boxplus (G'_j, S'_j))$, then $e = (u, v)(u', v)$ for $uu' \in E(G), v \in S'_j$ or $e = (u, v)(u, v')$ for $u \in S_i$ and $vv' \in E(G'_j)$. In the former case, $v \in S'_j$, thus $v \notin S'_p$, so $e \notin E(G_q, S_q) \boxplus (G'_p, S'_p)$. In the latter case $vv' \in E(G'_j)$, thus, since $E(G'_j) \cap E(G'_p) = \emptyset$, we have that $vv' \notin E(G'_p)$. Thus $e \notin (G_q, S_q) \boxplus (G'_p, S'_p)$. □

Proposition 7. *Let graphs G and G' each have a decomposition into $a \geq 1$ spanning subgraphs, $G_1, \dots, G_a$ and $G'_1, \dots, G'_a$, respectively. Then*

$$G \,\square\, G' = (G_1 \cup \cdots \cup G_a) \,\square\, (G'_1 \cup \cdots \cup G'_a) = \bigcup_{i=1}^{a} G_i \,\square\, G'_i,$$

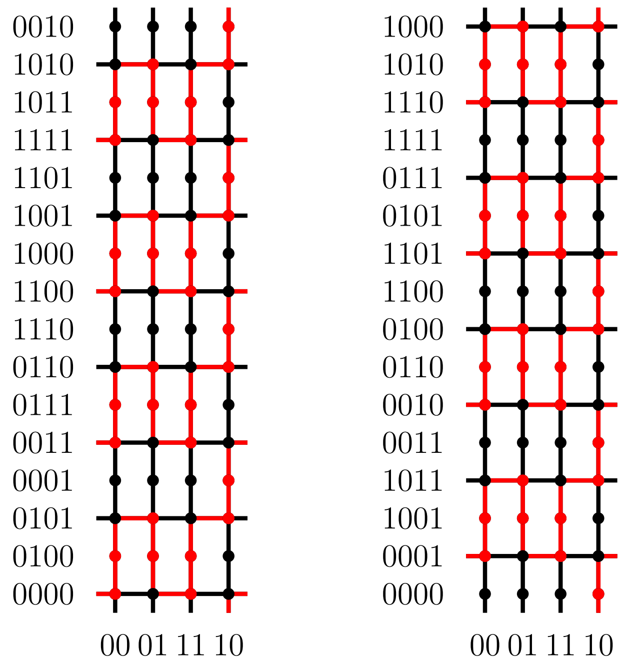

Figure 6: A 2-splittable decomposition of Q_6 into four cycles of the same length.

where the union is pairwise disjoint, i.e., a decomposition.

Proof. Consider an edge $e \in E(G \square G')$. Then $e = (u,v)(u',v)$ for $uu' \in E(G_i), v \in V(G')$ or $e = (u,v)(u,v')$ for $u \in V(G), vv' \in E(G'_i)$ for some $i = 1, \dots, k$. In both cases $e \in E(G_i \square G'_i)$. Clearly any edge in $G_i \square G'_i$ is in $G \square G'$. Assume that there is an edge e, $e \in E(G_i \square G'_i)$, $e \in E(G_j \square G'_j)$, $i \neq j$. Without loss of generality $e = (u,v)(u',v)$. Then $uu' \in E(G_i) \cap E(G_j)$, a contradiction. $\square$

Lemma 8. *Suppose the graph G has an (a,b)-DR-splittable decomposition into am cycles of the same even length and the graph G' has a c-splittable decomposition into cm cycles of the same length such that the representing sets in both decompositions have an even number of vertices. Then $G \square G'$ has a $2bc$-splittable decomposition into $2mac$ cycles of the same length, where all representing sets have an even number of vertices.*

Before giving the proof, we consider some examples: Figure 6 illustrates how Lemma 8 is applied to decompose Q_6 into 4 cycles. In this example, we write $Q_6 = Q_4 \square Q_2$, where Q_4 has a $(2,1)$-DR-splittable decomposition into two 16-cycles

$$C_1 = (0000, 0100, 0101, 0001, 0011, 0111, 0110, 1110, \\ 1100, 1000, 1001, 1101, 1111, 1011, 1010, 0010, 0000)$$

and

$$C_2 = (0000, 0001, 1001, 1011, 0011, 0010, 0110, 0100,$$
$$1100, 1101, 0101, 0111, 1111, 1110, 1010, 1000, 0000)$$

with representing sets

$$S_1 = \{0000, 0101, 0011, 0110, 1100, 1001, 1111, 1010\}$$

and

$$S_2 = \{0001, 1011, 0010, 0100, 1101, 0111, 1110, 1000\},$$

respectively. We know Q_2 has a 1-splittable decomposition into into one 4-cycle $(00, 01, 11, 10, 00)$. So $a = 2$, $b = 1$, $c = 1$, and $m = 1$, giving $2bc = 2$ and $2mac = 4$. Thus the result is a 2-splittable decomposition into 4 cycles. The two cycles in each subdivided torus split the vertices of Q_6, where vertex colors in the figure correspond to the representing sets in the resulting 2-splittable decomposition. Note that the vector corresponding to any vertex in Q_6 in the figure can be found by concatenating the vector to its left and the vector below.

Figure 7 illustrates how Lemma 8 is applied to decompose Q_6 into 8 cycles. Again, we write $Q_6 = Q_4 \square Q_2$, where Q_4 has a $(4, 2)$-DR-splittable decomposition into four 8-cycles

$$C_1 = (0000, 0100, 0101, 1101, 1111, 1011, 1010, 0010, 0000),$$
$$C_2 = (1100, 1000, 1001, 0001, 0011, 0111, 0110, 1110, 1100),$$
$$C_3 = (0100, 1100, 1101, 1001, 1011, 0011, 0010, 0110, 0100), \text{ and}$$
$$C_4 = (1000, 0000, 0001, 0101, 0111, 1111, 1110, 1010, 1000),$$

with representing sets

$$S_1 = \{0000, 0101, 1111, 1010\},$$
$$S_2 = \{1100, 1001, 0011, 0110\},$$
$$S_3 = \{0100, 1101, 1011, 0010\}, \text{ and}$$
$$S_4 = \{1000, 0001, 0111, 1110\},$$

respectively. In this decomposition we take $\mathcal{F}_1 = \{C_1, C_2, C_3, C_4\}$, with $\mathcal{F}_{1,1} = \{C_1, C_2\}$ and $\mathcal{F}_{1,2} = \{C_3, C_4\}$, i.e. $V(Q_4)$ is partitioned by $S_1 \cup S_2 \cup S_3 \cup S_4$, and also by $V(C_1) \cup V(C_2)$ and $V(C_3) \cup V(C_4)$. We know Q_2 has a 1-splittable decomposition into into one 4-cycle $(00, 01, 11, 10, 00)$. So $a = 4$, $b = 2$, $c = 1$, and $m = 1$, giving $2bc = 4$, and $2mac = 8$. Thus the result is a 4-splittable decomposition into 8 cycles. The vertex colors correspond to the representing sets in the resulting 4-splittable decomposition, where the four cycles from the left two tori and the four cycles from the right two tori each split the vertices of Q_6.

Proof. Let $C_1, \ldots, C_{am}$ and $C'_1, \ldots, C'_{cm}$ be the cycles decomposing G and G', respectively, with representing sets $S_1, \ldots, S_{am}$ and $S'_1, \ldots, S'_{cm}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be splitting sets, with splitting subsets $\mathcal{F}_{i,1}, \ldots, \mathcal{F}_{i,a/b}$ for $1 \leq i \leq m$, of the (a, b)-splittable decomposition of G, and $\mathcal{F}'_1, \cdots, \mathcal{F}'_m$ be the splitting sets for the c-splittable decomposition of G'. That is, for $i = 1, \ldots, m$, $\mathcal{F}_i$ consists of a cycles C_s, and can be partitioned into subsets $\mathcal{F}_{i,1}, \ldots, \mathcal{F}_{i,a/b}$ where the b cycles in each $\mathcal{F}_{i,j}$ are vertex disjoint and span $V(G)$. Similarly, $\mathcal{F}'_i$ consists of c cycles C'_t, and $\mathcal{F}'_i$ forms a splittable decomposition of a spanning subgraph of G', $i = 1, \ldots, m$. Then

$$
\begin{aligned}
G \,\square\, G' &= \bigcup_{i=1}^{m} \bigcup_{C \in \mathcal{F}_i} C \quad \square \quad \bigcup_{i=1}^{m} \bigcup_{C' \in \mathcal{F}'_i} C' \\
&= \bigcup_{i=1}^{m} \left(\bigcup_{C \in \mathcal{F}_i} C \quad \square \quad \bigcup_{C' \in \mathcal{F}'_i} C' \right) && \text{by Proposition 7} \\
&= \bigcup_{i=1}^{m} \bigcup_{C_s \in \mathcal{F}_i} \bigcup_{C'_t \in \mathcal{F}'_i} (C_s, S_s) \boxplus (C'_t, S'_t) && \text{by Proposition 6.}
\end{aligned}
$$

Each $(C_s, S_s) \boxplus (C'_t, S'_t)$ is a subdivided torus, denote it by $T_{s,t}$. Recall that these tori are pairwise edge-disjoint (see Proposition 6) and the unions are pairwise edge-disjoint (see Proposition 7). Since each $|S_s|$ and $|S'_t|$ is even, $T_{s,t}$ allows the 2-wiggle decomposition, and decomposes into two cycles, $C_{s,t}$ and $C'_{s,t}$. Since each (C_s, S_s) is distance regular, by Proposition 2, $C_{s,t}$ and $C'_{s,t}$ have same length. This gives a decomposition of $G \,\square\, G'$ into $2 \cdot m \cdot a \cdot c$ cycles. Since each S_s has the same cardinality, and each S'_t has the same cardinality, all tori $T_{s,t}$ have the same number of edges and thus all the resulting cycles of the decomposition have the same length.

We need to argue that the resulting cycle decomposition is $2bc$-splittable, i.e., the cycles can be grouped into splitting sets of size $2bc$ each, where each cycle has a representing set of the same even cardinality, and the representing sets for a given splitting set partition $V(G \,\square\, G')$. For $i = 1, \ldots, m$, $j = 1, \ldots, a/b$, let the splitting set $\mathcal{H}_{i,j} = \{C_{s,t}, C'_{s,t} : C_s \in \mathcal{F}_{i,j}, C'_t \in \mathcal{F}'_i\}$. Note that each $\mathcal{H}_{i,j}$ contains $2bc$ cycles, and each cycle in the decomposition is in exactly one such set. It remains to assign representing sets of even cardinality to each cycle in $\mathcal{H}_{i,j}$ so that they partition $V(G \,\square\, G)$.

Fix i and j. Given $C_s \in \mathcal{F}_{i,j}$ and $C'_t \in \mathcal{F}'_i$ we will split the vertices in each $V(C_s) \times S'_t$ into two sets $S_{s,t}$ and $S'_{s,t}$ to form representing sets for $C_{s,t}$ and $C'_{s,t}$. First we verify that this will partition the vertices in $V(G \,\square\, G')$. Since the sets $\{V(C_s) : C_s \in \mathcal{F}_{i,j}\}$ partition $V(G)$, for a given t, the sets $\{V(C_s) \times S'_t : \; C_s \in \mathcal{F}_{i,j}\}$ partition $V(G) \times S'_t$. Since the sets $\{S'_t : \; C'_t \in \mathcal{F}'_i\}$ partition $V(G')$, the set $\{V(C_s) \times S'_t : \; C_s \in \mathcal{F}_{i,j}, C'_t \in \mathcal{F}'_i\}$ partitions $V(G) \times V(G') = V(G \,\square\, G')$.

Since (C_s, S_s) is distance regular, Proposition 5 assures that we can find $S_{s,t} \subseteq V(C_{s,t})$ and $S'_{s,t} \subseteq V(C'_{s,t})$ where these sets have the same cardinality and partition $V(C_s) \times S'_t$. Further, since every C_s is of the same even length and every S'_t has the same even cardinality, for every $C_s \in \mathcal{F}_{i,j}, C'_t \in \mathcal{F}'_i$, the set $V(C_s) \times S'_t$ contains the same number of vertices, and this number is a multiple of four. This implies the number of vertices in $S_{s,t}$ and $S'_{s,t}$ is even. $\square$

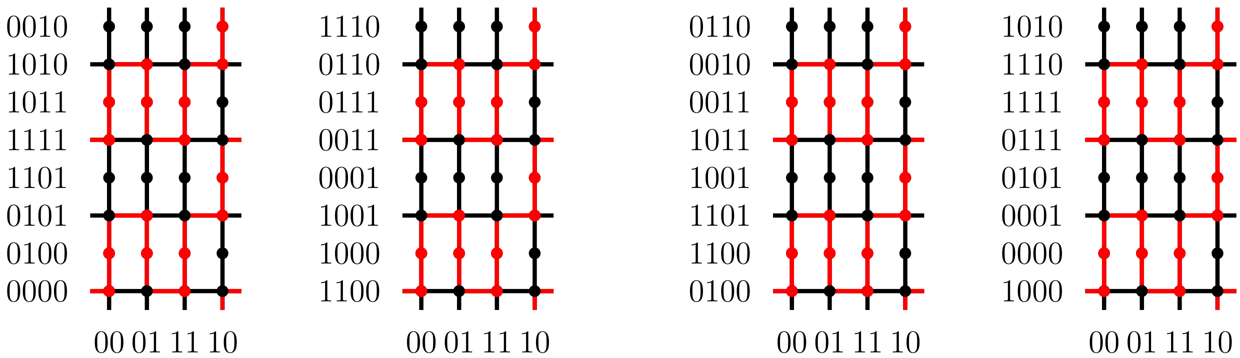

Figure 7: A 4-splittable decomposition of Q_6 into eight cycles of the same length.

4.1 Decomposition of products without increasing cycle length

In this subsection, we prove, under two different splittability conditions, two propositions which imply that if G has a decomposition into cycles of a given length, then $G \,\square\, G$ has a decomposition into cycles of the same length.

Proposition 9. *If G has an (a,b)-DR-splittable cycle decomposition into cycles of length ℓ, then $G \,\square\, G$ has an $(a|V(G)|, b|V(G)|)$-DR-splittable cycle decomposition into cycles of length ℓ.*

Proof. Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be the splitting sets of the (a,b)-DR-splittable cycle decomposition of G, with splitting subsets $\mathcal{F}_{i,1}, \ldots, \mathcal{F}_{i,a/b}$ for $1 \le i \le m$. Recall the definition of vertical and horizontal graphs and cycles given in Section 3. The product $G \,\square\, G$ can be decomposed into $2|V(G)|$ edge-disjoint copies of G: $|V(G)|$ horizontal copies induced by $\{(u,v) : v \in V(G)\}$ for a fixed $u \in V(G)$, and $|V(G)|$ vertical copies induced by $\{(u,v) : u \in V(G)\}$ for a fixed $v \in V(G)$. Copy the cycle decomposition of G into each of these copies to obtain a cycle decomposition of $G \,\square\, G$. For $1 \le i \le m$, let $\mathcal{F}'_i$ consist of all images of the cycles in the splitting set $\mathcal{F}_i$ in the horizontal cycles. Then $\mathcal{F}'_i$ contains $a|V(G)|$ cycles. For representing sets, assign to each cycle the image of its representing set from the decomposition of G. Since the representing sets in $\mathcal{F}_i$ partition $V(G)$, the representing sets in $\mathcal{F}'_i$ partition $V(G \,\square\, G)$, and are still distance regular. For $1 \le i \le m$, $i \le j \le a/b$, let the splitting subset $\mathcal{F}'_{i,j}$ contain the image of all cycles from $\mathcal{F}_{i,j}$ in the horizontal copies of G. Note that each $\mathcal{F}'_{i,j}$ contains $b|V(G)|$ cycles and the vertices in these cycles partition $V(G \,\square\, G)$. Doing the same thing with the vertical copies of G creates more splitting sets $\mathcal{F}''_i$, with splitting subsets $\mathcal{F}''_{i,j}$, and together all of the splitting sets $\mathcal{F}'_i$ and F''_i with splitting subsets $\mathcal{F}'_{i,j}$ and $\mathcal{F}''_{i,j}$ give an $(a|V(G)|, b|V(G)|)$-DR-splittable cycle decomposition of $G \,\square\, G$ into cycles of length ℓ. □

Proposition 10. *Let G be a graph with an (a,b)-DR-splittable decomposition into cycles of length ℓ, where $|V(G)|/a$ is even and greater than two. Then $G \,\square\, G$ has a $(2a|V(G)|, b|V(G)|)$-DR-splittable decompositon into cycles of length ℓ, where each representing set has cardinality at least two.*

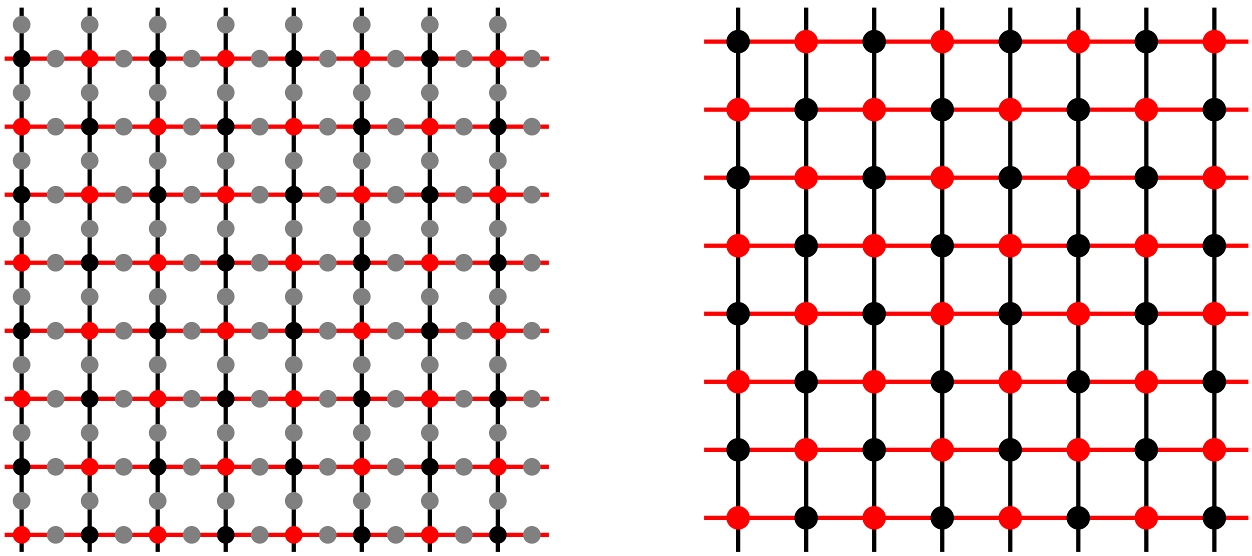

Figure 8: Left: How to split the vertices in Proposition 10. Right: The split of the vertices on the underlying torus. In both cases, the black vertices are the representing sets for the black vertical cycles, and the red vertices are the representing sets for the red horizontal cycles.

Proof. Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be the splitting sets of of the (a, b)-DR-splittable decomposition of G. First we shall only use the property that this decomposition is a-DR-splittable. Let G_i denote the union of graphs in $\mathcal{F}_i$, and note that each G_i is a spanning subgraph of G. By Proposition 7, $G \,\square\, G$ can be decomposed as $G \,\square\, G = \bigcup_{i=1}^{m} G_i \,\square\, G_i$.

We now focus on decomposing each of the products $G_i \,\square\, G_i$ in the union. Let $\mathcal{F}_i = \{C_1, \ldots, C_a\}$, with representative sets $S_1, \ldots, S_a$. By Proposition 6, $G_i \,\square\, G_i$ can be decomposed as $G_i \,\square\, G_i = \bigcup_{C_s \in \mathcal{F}_i} \bigcup_{C_t \in \mathcal{F}_i} (C_s, S_s) \boxplus (C_t, S_t)$.

Since $|S_s| = |V(G)|/a$, for $s = 1, \ldots, a$, each of the a^2 subdivided tori $(C_s, S_s) \boxplus (C_t, S_t)$ has $|V(G)|/a$ vertical cycles and $|V(G)|/a$ horizontal cycles, each of length ℓ. We choose the set $\mathcal{F}'_i$ of all of the horizontal and vertical cycles in all a^2 subdivided tori as our decomposition of $G_i \,\square\, G_i$, and thus $|\mathcal{F}'_i| = 2(|V(G)|/a)a^2 = 2a|V(G)|$, $i = 1, \ldots, m$. We now assign representative sets as illustrated in Figure 8 (left): Each vertex in $V(G_i \,\square\, G_i) = V(G \,\square\, G)$ appears once as a degree 4 vertex in exactly one of the subdivided tori $(C_s, S_s) \boxplus (C_t, S_t)$. Thus to assign each vertex in $G \,\square\, G$ to exactly one representing set, we only assign to a given cycle degree four vertices from its subdivided torus, and we can instead focus on the underlying torus, as shown in Figure 8 (right). In the underlying torus, properly two-color the vertices red and black, assigning the red vertices to be the representing sets of the horizontal cycle that they are on, and assigning the black vertices to be the representing sets for the vertical cycles they are on. Since there is only one proper two-coloring, and this coloring alternates red and black on every horizontal and vertical cycle, each representing set is the same cardinality. Further, since every other vertex is chosen, in the subdivided torus, these representing sets split the cycles from $\mathcal{F}'_i$ into paths twice as long as the corresponding paths on cycles in $\mathcal{F}_i$ with the original representing sets. This shows that the resulting decomposition with splitting sets $\mathcal{F}'_1, \ldots, \mathcal{F}'_m$ is a $2a|V(G)|$-DR-splittable decomposition of $G \,\square\, G$. Note we need $|V(G)|/a > 2$ so that every cycle in $\mathcal{F}'_i$ has at least 2 vertices in its representing set.

Now that we have a $2a|V(G)|$-DR-splittable decomposition of $G \,\square\, G$ with splitting sets $\mathcal{F}'_1, \ldots, \mathcal{F}'_m$, we show that it is also a $(2a|V(G)|, b|V(G)|)$-DR-splittable decompositon. Since $\mathcal{F}_1, \ldots, \mathcal{F}_m$ are splitting sets of an (a, b)-DR-splittable decomposition, each family $\mathcal{F}_i$ can be partitioned into splitting subsets $\mathcal{F}_{i,j}$, each consisting of $b = |V(G)|/\ell$ cycles in $\mathcal{F}_i$ that are pairwise vertex disjoint and span $V(G)$, $j = 1, \ldots, a/b$.

For $1 \leq j \leq a/b$, let $\mathcal{F}(V)'_{i,j}$ be all of the vertical cycles in the subdivided tori

$$\bigcup_{C_s, \in \mathcal{F}_{i,j}} \bigcup_{C_t \in \mathcal{F}_i} (C_s, S_s) \boxplus (C_t, S_t)$$

and let $\mathcal{F}(H)'_{i,j}$ be all of the horizontal cycles in the subdivided tori

$$\bigcup_{C_s, \in \mathcal{F}_i} \bigcup_{C_t \in \mathcal{F}_{i,j}} (C_s, S_s) \boxplus (C_t, S_t).$$

For all i and j, $\mathcal{F}(V)'_{i,j}$ contains a vertical copy of every cycle in $\mathcal{F}_{i,j}$ for every vertex in G. Thus it contains $b|V(G)|$ cycles, and these cycles partition the vertices of $G \,\square\, G$. Similarly, $\mathcal{F}(H)'_{i,j}$ contains a horizontal copy of every cycle in $\mathcal{F}_{i,j}$ for every vertex in G. Thus it contains $b|V(G)|$ cycles, and these cycles partition the vertices of $G \,\square\, G$. Finally, the union of all such sets is $\mathcal{F}'_i$, so the $\mathcal{F}(V)'_{i,j}$ and $\mathcal{F}(H)'_{i,j}$ are the required splitting subsets. $\square$

5 Proof of Theorem 1

First we shall prove a result about hypercube decompositions into cycles whose lengths are powers of 2. Then, we prove the main theorem and its corollary.

Lemma 11. *Let $x \geq 1$ be odd. For integers $n \geq 1$ and $\ell \geq 2$ where $2x \leq \ell \leq x2^n$, Q_{x2^n} has a $(2^m, 2^{x2^n - \ell})$-DR-splittable decomposition into cycles of length 2^ℓ for each m,*

$$x2^n - \ell \leq m \leq \min\{x2^n - 1, x2^n - 1 + n - \ell\}.$$

Proof. Let x be an odd positive integer. We have to prove the statement of the lemma for pairs (ℓ, n) in the allowed range. These pairs are pictured as dots in Figure 9, which contains a visualization of the order in which the cases are proved in the case $x = 1$. First we shall prove a claim that the lemma is true for pairs (ℓ, n) when $x2^{n-1} < \ell \leq x2^n$. These are the cases pictured as black dots in Figure 9.

Claim. For any $n \geq 1$ the following holds: if $\ell \geq 2x$ and $x2^{n-1} < \ell \leq x2^n$, then Q_{x2^n} has a $(2^m, 2^{x2^n-\ell})$-DR-splittable decomposition into cycles of length 2^ℓ for any m such that $x2^n - \ell \leq m \leq \min\{x2^n - 1, x2^n - 1 + n - \ell\}$.

We shall prove the claim by induction on n. Note that here $\min\{x2^n-1, x2^n-1+n-\ell\} = x2^n - 1 + n - \ell$ since $n < \ell$.

Base case $n = 1$. If $n = 1$ then we must have $\ell = 2x$. Note that $x2^1 - \ell = 2x - 2x = 0$, and $x2^1 - 1 + 1 - \ell = 2x - 1 + 1 - 2x = 0$, so we seek a $(2^0, 2^0) = (1,1)$-DR-splittable decomposition of $Q_{x2^1} = Q_{2x}$. By the result of Aubert and Schneider [3] Q_{2x} has a Hamiltonian decomposition into cycles of length 2^{2x}, which is a $(1,1)$-DR-splittable decomposition of Q_{2x}.

Assume the statement is true for some n, and consider ℓ such that $\ell \geq 2x$ and $x2^n < \ell \leq x2^{n+1}$. By the inductive hypothesis, Q_{x2^n} has an (a,b)-DR-splittable cycle decomposition for $b = 1$ and $a = 2^{m'}$ for all $0 \leq m' \leq n-1$. Note that since $b = 1$, all cycles in this decomposition are Hamiltonian, with length 2^{x2^n}. Pick $0 \leq m' \leq n-1$, and suppose the splitting sets of cycles in the $(2^{m'}, 1)$-DR-splittable decomposition of Q_{x2^n} are $\mathcal{F}_1, \ldots, \mathcal{F}_{x2^{n-1-m'}}$. (Note that since $b = 1$, the splitting subsets $\mathcal{F}_{i,j}$ contain one cycle each). Then by Proposition 7,

$$Q_{x2^{n+1}} = Q_{x2^n} \,\Box\, Q_{x2^n} = \bigcup_{i=1}^{x2^{n-1-m'}} \bigcup_{C \in \mathcal{F}_i} C \,\Box\, C.$$

This gives a decomposition of $Q_{x2^{n+1}}$ into $x2^{n-1}$ tori $C \,\Box\, C$, each with $2 \cdot 2^{x2^n} \cdot 2^{x2^n} = 2^{x2^{n+1}+1}$ edges. Thus for our given ℓ, letting $k = 2^{x2^{n+1}-\ell+1}$ (Since $x2^n < \ell \leq x2^{n+1}$, k could take any value of $2^{k'}$ where $1 \leq k' \leq x2^n$), each torus allows the k-wiggle decomposition, which results in each torus being decomposed into k cycles, each with length 2^ℓ.

Now we show the decomposition produced by applying the k-wiggle decomposition to each torus is $(2^m, 2^{x2^{n+1}-\ell})$-DR-splittable for all values of m where $x2^{n+1} - \ell \leq m \leq x2^{n+1} - 1 + (n+1) - \ell$. Let $\mathcal{F}'_i$ be the set of $k2^{m'}$ cycles decomposing the tori $\bigcup_{C \in \mathcal{F}_i} C \,\Box\, C$. Since the horizontal cycles in the tori have distance regular representing sets, Proposition 4 guarantees that the $k2^{m'}$ cycles in $\mathcal{F}'_i$ yielded by the decomposition of the tori generated by a splitting set $\mathcal{F}_i$ are $k2^{m'}$-DR-splittable. For the values $0 \leq m' \leq n-1$, the values of $k2^{m'}$ take on any value of 2^m where $x2^{n+1} - \ell + 1 \leq m \leq x2^{n+1} - 1 + (n+1) - \ell$.

To complete the claim, we need to show this decomposition is also $2^{x2^{n+1}-\ell}$-DR-splittable. Since all choices of k we consider are even, Proposition 3 guarantees that the set of cycles decomposing each torus in $C \,\Box\, C$ is $k/2 = 2^{x2^{n+1}-\ell}$-splittable, where the representing sets for each cycle contain all vertices of the cycle. Let the splitting sets $\mathcal{F}'_i$ each be a set of $k/2$ cycles given by Proposition 3 that partition the vertices of $C \,\Box\, C$. Since the distance between consecutive vertices in the representing sets is 1, we obtain a $2^{x2^{n+1}-\ell}$-DR-splittable decomposition. Note that the splitting sets $\mathcal{F}'_i$ in this decomposition partition the splitting sets of every other one, since the splitting sets of cycles in each other splittable decomposition are composed of all cycles from one or more tori. Thus these $\mathcal{F}'_i$ can serve as the splitting subsets for the other decompositions, and we have a $(2^m, 2^{x2^{n+1}-\ell})$-DR-splittable decomposition for every $x2^{n+1}-\ell \leq m \leq x2^{n+1}-1+(n+1)-\ell$. This completes the proof of the claim.

Now, we shall prove the statement of the lemma. Fix an integer ℓ, $\ell \geq 2x$. Let n be a positive integer such that $2x \leq \ell \leq x2^n$. Let n' be a positive integer such that $x2^{n'-1} < \ell \leq x2^{n'}$. We see that $n \geq n'$. We shall prove the statement of the proposition by induction on $n-n'$. If $n-n'=0$, i.e., $n=n'$, we are done by the claim. Assume that the statement of the lemma holds for $n \geq n'$, i.e. Q_{x2^n} has a $(2^m, 2^{x2^n-\ell})$-DR-splittable decomposition into cycles of length 2^ℓ for every $x2^n-\ell \leq m \leq x2^n-1+n-\ell$. We now prove the statement for $n+1$.

Case 1. $n<\ell$. These cases are represented by the blue dots in Figure 9. Since $Q_{x2^{n+1}} = Q_{x2^n} \,\square\, Q_{x2^n}$ and $|V(Q_{x2^n})| = 2^{x2^n}$, applying Proposition 9 with $a=2^{m'}$ for $x2^n-\ell \leq m' \leq x2^n-1+n-\ell$ and $b=2^{x2^n-\ell}$ gives a (a',b')-DR-splittable decomposition where $b'=2^{x2^n-\ell}2^{x2^n}=2^{x2^{n+1}-\ell}$, and a' can be 2^m for any value of m from $(x2^n-\ell)+x2^n = x2^{n+1}-\ell$ to $(x2^n-1+n-\ell)+x2^n = x2^{n+1}-2+(n+1)-\ell$. It remains to show that $Q_{x2^{n+1}}$ has a $(2^{x2^{n+1}-1+(n+1)-\ell}, 2^{x2^{n+1}-\ell})$-DR-splittable decomposition. Applying Proposition 10 to $Q_{x2^{n+1}} = Q_{x2^n} \,\square\, Q_{x2^n}$ with $a=2^{x2^n-1+n-\ell}$, $b=2^{x2^n-\ell}$, and $|V(G)|=|V(Q_{x2^n})|=2^{x2^n}$, we get an (a',b')-DR-splittable decomposition with

$$a' = 2a|V(Q_{x2^n})| = 2\cdot 2^{x2^n-1+n-\ell}\cdot 2^{x2^n} = 2^{x2^{n+1}-1+(n+1)-\ell}$$

and

$$b' = b|V(Q_{x2^n})| = 2^{x2^{n+1}-\ell}.$$

Case 2. $n \geq \ell$. These cases are represented by the red dots in Figure 9, and follow from applying Proposition 9 exactly as in Case 1. Since $n \geq \ell$, in this case $\min\{x2^n-1, x2^n-1+n-\ell\} = x2^n-1$, so Proposition 10 is not needed. $\square$

We are now ready to prove the main theorem. We actually prove the following stronger statement: Let $n = xy2^\alpha$, where $x,y \geq 1$ are odd, and $\alpha \geq 1$. Suppose y has binary representation $y = 2^{i_1}+2^{i_2}+\cdots+2^{i_j}$, where $i_1 > i_2 > \cdots > i_j = 0$. Then for $0 \leq q \leq n-i_1-2xj$, Q_n has a 2^{j-1+q}-splittable decomposition into $x2^{i_1+\alpha+j-2+q}$ cycles of the same length.

Proof of Theorem 1. We shall use induction on j.

Base case $j=1$. If $j=1$, then $y=2^0=1$, so $i_1=0$ and $n=x2^\alpha$, where $\alpha \geq 1$. Lemma 11 implies that Q_{x2^α} has a $2^{x2^\alpha-\ell}$-splittable decomposition into $x2^{x2^\alpha-1+\alpha-\ell}$ cycles of length 2^ℓ for each $2x \leq \ell \leq x2^\alpha$. Assigning ℓ all values in the range from $2x$ to $x2^\alpha$ gives all required decompositions, from a $2^{x2^\alpha-\ell} = 2^{x2^\alpha-2x} = 2^{j-1+(x2^\alpha-i_1-2xj)}$-splittable decomposition into $x2^{x2^\alpha-1+\alpha-\ell} = x2^{x2^\alpha-1+\alpha-2x} = x2^{i_1+\alpha+j-2+(x2^\alpha-i_1-2xj)}$ cycles when $\ell=2x$, to a $2^{x2^\alpha-\ell}=2^0=2^{j-1+0}$-splittable decomposition into $x2^{x2^\alpha-1+\alpha-\ell} = x2^{\alpha-1} = x2^{i_1+\alpha+j-2+0}$ cycles when $\ell = x2^\alpha$.

Inductive step: Let $n = xy2^\alpha = x(2^{i_1}+2^{i_2}+\cdots+2^{i_j})2^\alpha$ with $j>1$. Then $Q_n = Q_{x2^{i_1+\alpha}} \,\square\, Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$, so we seek to apply Lemma 8 with $G = Q_{x2^{i_1+\alpha}}$ and $G' = Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$.

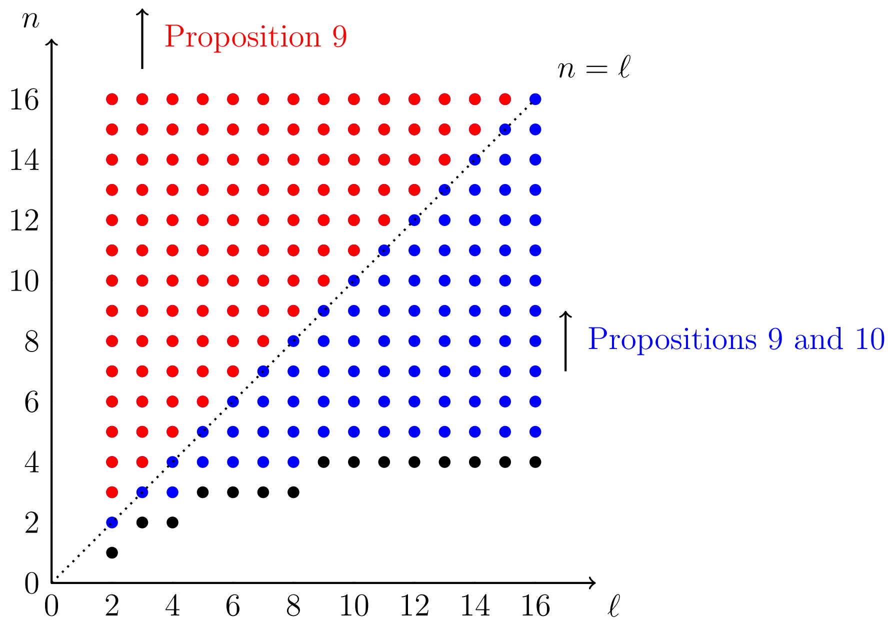

Figure 9: Schematic for proof of Lemma 11 in the case $x = 1$. The lemma in the cases (ℓ, n) represented by the black dots are proved in the initial claim. Then for a given (ℓ, n) where $n < \ell$ where Lemma 11 holds, Propositions 9 and 10 are used in the induction to prove the lemma in the case $(\ell, n+1)$ (blue dots below the line $n = \ell$). Finally, for a given (ℓ, n) where $n \geq \ell$ where Lemma 11 holds, only Proposition 9 is needed in the induction to prove the lemma in the case $(\ell, n+1)$ (red dots above the line $n = \ell$).

By Lemma 11, $Q_{x2^{i_1+\alpha}}$ has a $(2^{x2^{i_1+\alpha}-i+Z}, 2^{x2^{i_1+\alpha}-i})$-DR-splittable decomposition into $x2^{x2^{i_1+\alpha}-1+i_1+\alpha-\ell}$ cycles, where $2x \leq \ell \leq x2^{i_1+\alpha}$ and $0 \leq Z \leq \min\{\ell-1, i_1+\alpha-1\}$. We will choose $Z = i_1 - i_2$ and thus for the remainder of the proof we will enforce the restriction that $2x + (i_1 - i_2) \leq \ell$, simultaneously ensuring that $2x \leq \ell$ and $Z = i_1 - i_2 \leq \ell - 1$.

By the inductive hypothesis, $Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$ has a 2^{j-2+q}-splittable decomposition into $x2^{i_2+\alpha+j-3+q}$ cycles, where $0 \leq q \leq x(2^{i_2}+\cdots+2^{i_j})2^\alpha - i_2 - 2x(j-1)$.

Let $c = 2^{j-2+q}$, $m = x2^{i_2+\alpha-1}$, $a = 2^{x2^{i_1+\alpha}+i_1-i_2-\ell}$, and $b = 2^{x2^{i_1+\alpha}-\ell}$. Then $Q_{x2^{i_1+\alpha}}$ has an (a,b)-DR-splittable decomposition into am cycles, and $Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$ has a c-splittable decomposition into cm cycles. Since $\ell > i_1 - i_2$, $a = 2^{x2^{i_1+\alpha}+i_1-i_2-\ell}$ divides $|V(Q_{x2^{i_1+\alpha}})| = 2^{x2^{i_1+\alpha}}$ with even quotient, so the representing sets in the decomposition of $Q_{x2^{i_1+\alpha}}$ have even cardinality at least two. Similarly, since

$$c = 2^{j-2+q} \leq 2^{j-2+x(2^{i_2}+\cdots+2^{i_j})2^\alpha - i_2 - 2x(j-1)} \leq 2^{x(2^{i_2}+\cdots+2^{i_j})2^\alpha-(2x-1)(j-1)-1} < 2^{x(2^{i_2}+\cdots+2^{i_j})2^\alpha},$$

c divides $|V(Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha})| = 2^{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$ with even quotient, so the representing sets in the decomposition of $Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$ have even cardinality at least two. Thus we can apply Lemma 8 with $G = Q_{x2^{i_1+\alpha}}$ and $G' = Q_{x(2^{i_2}+\cdots+2^{i_j})2^\alpha}$ to obtain a $2bc$-splittable decomposition into $2mac$ cycles. Here

$$2bc = 2 \cdot 2^{x2^{i_1+\alpha}-\ell} \cdot 2^{j-2+q} = 2^{x2^{i_1+\alpha}-\ell+j-1+q}$$

and

$$2mac = 2 \cdot x2^{i_2+\alpha-1} \cdot 2^{x2^{i_1+\alpha}+i_1-i_2-\ell} \cdot 2^{j-2+q} = x2^{x2^{i_1+\alpha}+\alpha-\ell+i_1+j-2+q}.$$

Letting the parameters ℓ and q range over $2x + i_1 - i_2 \leq \ell \leq x2^{i_1+\alpha}$ and $0 \leq q \leq x(2^{i_2}+\cdots+2^{i_j})2^\alpha - i_2 - 2x(j-1)$ gives

$$2^{j-1+0} \leq 2bc \leq 2^{j-1+(n-i_1-2xj)}$$

and

$$x2^{\alpha+i_1+j-2+0} \leq 2mac \leq x2^{\alpha+i_1+j-2+(n-i_1-2xj)}.$$

The lower bounds are obtained when $\ell = x2^{i_1+\alpha}$ and $q = 0$, while the upper bounds are obtained when $\ell = 2x + i_1 - i_2$ and $q = x(2^{i_2}+\cdots+2^{i_j})2^\alpha - i_2 - 2x(j-1)$. □

Proof of Corollary 1. Letting $x = 1$ and $q = 0$ in Theorem 1 gives a decomposition of Q_n into cycles of length $\ell = y2^n2^{1-i_1-j}$, where $n = y2^\alpha = (2^{i_1}+\cdots+2^{i_j})2^\alpha$, $i_1 > i_2 > \cdots > i_j = 0$. Since i_1 and j are each at most $\log_2 y$, we see that $2^{i_1+j} \leq y^2$. Thus $\ell \geq y2^{n+1}/y^2 = 2^{n+1}/y \geq 2^{n+1}/n$. Further note that ℓ is divisible by y, the largest odd divisor of n.

Having a decomposition into cycles of length ℓ, we can split each cycle into paths of length ℓ' as long as ℓ' divides ℓ, i.e., $\ell' = y'2^{m'}$ for y' an odd divisor of y and some m', $\ell' \leq \ell/2 \leq 2^n/n$. The number of edges in Q_n is $n2^{n-1} = y2^m$, for an integer m. The corollary follows from the fact that any number dividing the number of edges in Q_n has a form $y'2^{m'}$, for y' an odd divisor of y and some m'. □

Finally, we note that in the case $x = 1$ it is possible to make a slightly stronger statement than Theorem 1, which we prove here, along with a corollary.

Proposition 12. *Let n be even, with binary representation $n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_j}$, where $i_1 > i_2 > \cdots > i_j$. Then for $0 \leq q \leq n - i_1 - j$, Q_n has a 2^{j-1+q}-splittable decomposition into $2^{i_1+j-2+q}$ cycles of the same length.*

Proof. We proceed by induction on j.

Base case $j = 1$. If $j = 1$, then $n = 2^{i_1}$, where $i_1 \geq 1$. Lemma 11 implies that $Q_{2^{i_1}}$ has a $2^{2^{i_1}-\ell}$-splittable decomposition into $2^{2^{i_1}-1+i_1-\ell}$ cycles when $2 \leq \ell \leq 2^{i_1}$. Assigning ℓ all values in the range from $i_1 + 1$ to 2^{i_1} gives all required decompositions, from a $2^{2^{i_1}-\ell} = 2^{2^{i_1}-i_1-1} = 2^{j-1+(2^{i_1}-i_1-j)}$-splittable decomposition into $2^{2^{i_1}-1+i_1-\ell} = 2^{2^{i_1}-2} = 2^{i_1+j-2+(2^{i_1}-i_1-j)}$ cycles when $\ell = i_1 + 1$, to a $2^{2^{i_1}-\ell} = 2^0 = 2^{j-1+0}$-splittable decomposition into $2^{2^{i_1}-1+i_1-\ell} = 2^{i_1-1} = 2^{i_1+j-2+0}$ cycles when $\ell = 2^{i_1}$.

Inductive step: Let $n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_j}$, with $j > 1$. Then $Q_n = Q_{2^{i_1}} \,\Box\, Q_{2^{i_2}+\cdots+2^{i_j}}$, so we seek to apply Lemma 8 with $G = Q_{2^{i_1}}$ and $G' = Q_{2^{i_2}+\cdots+2^{i_j}}$.

By Lemma 11 , $Q_{2^{i_1}}$ has a $(2^{2^{i_1}-\ell+Z}, 2^{2^{i_1}-\ell})$-DR-splittable decomposition into $2^{2^{i_1}-1+i_1-\ell}$ cycles, where $2 \leq \ell \leq 2^{i_1}$ and $0 \leq Z \leq \min\{\ell - 1, i_1 - 1\}$. We will choose $Z = i_1 - i_2$ and thus for the remainder of the proof we have the restriction $i_1 - i_2 + 1 \leq \ell$, ensuring $2 \leq \ell$ and $Z \leq \ell - 1$.

By the inductive hypothesis, $Q_{2^{i_2}+\cdots+2^{i_j}}$ has a 2^{j-2+q}-splittable decomposition into $2^{i_2+j-3+q}$ cycles, where $0 \leq q \leq 2^{i_2} + \cdots + 2^{i_j} - i_2 - j + 1$.

Let $c = 2^{j-2+q}$, $m = 2^{i_2-1}$, $a = 2^{2^{i_1}+i_1-i_2-\ell}$, and $b = 2^{2^{i_1}-\ell}$. Then $Q_{2^{i_1}}$ has an (a, b)-DR-splittable decomposition into am cycles, and $Q_{2^{i_2}+\cdots+2^{i_j}}$ has a c-splittable decomposition into cm cycles. Since $\ell > i_1 - i_2$, $a = 2^{2^{i_1}+i_1-i_2-\ell}$ divides $|V(Q_{2^{i_1}})| = 2^{2^{i_1}}$ with even quotient, so the representing sets in the decomposition of $Q_{2^{i_1}}$ have even cardinality at least two. Similarly, since $q < 2^{i_2} + \cdots + 2^{i_j} - j + 2$, $c = 2^{j-2+q}$ divides $|V(Q_{2^{i_2}+\cdots+2^{i_j}})| = 2^{2^{i_2}+\cdots+2^{i_j}}$ with even quotient, so the representing sets in the decomposition of $Q_{2^{i_2}+\cdots+2^{i_j}}$ have even cardinality at least two. Thus we can apply Lemma 8 with $G = Q_{2^{i_1}}$ and $G' = Q_{2^{i_2}+\cdots+2^{i_j}}$ to obtain a $2bc$-splittable decomposition into $2mac$ cycles. Here

$$2bc = 2 \cdot 2^{2^{i_1}-\ell} \cdot 2^{j-2+q} = 2^{2^{i_1}-\ell+j-1+q}$$

and

$$2mac = 2 \cdot 2^{i_2-1} \cdot 2^{2^{i_1}+i_1-i_2-\ell} \cdot 2^{j-2+q} = 2^{2^{i_1}-\ell+i_1+j-2+q}.$$

Letting the parameters ℓ and q range over $i_1 - i_2 + 1 \leq \ell \leq 2^{i_1}$ and $0 \leq q \leq 2^{i_2} + \cdots + 2^{i_j} - i_2 - j + 1$ gives

$$2^{j-1+0} \leq 2bc \leq 2^{j-1+(n-i_1-j)}$$

and

$$2^{i_1+j-2+0} \leq 2mac \leq 2^{i_1+j-2+(n-i_1-j)}.$$

The lower bounds are obtained when $\ell = x2^{i_1}$ and $q = 0$, while the upper bounds are obtained when $\ell = i_1 - i_2 + 1$ and $q = 2^{i_2} + \cdots + 2^{i_j} - i_2 - j + 1$. □

The following corollary shows that we get a decomposition of Q_n into almost all cycles whose length divides $n2^{n-1}$ and is divisible by $2n$.

Corollary. *Let n be even. Then there is a decomposition of Q_n into cycles of length $n2^m$ if $m \geq 1$ and $n2^m \leq 2^n/n$.*

Proof. By Proposition 12, Q_n can be decomposed into $2^{i_1+j-2+q}$ cycles of the same length, where $0 \leq q \leq n-i_1-j$. Since Q_n has $n2^{n-1}$ edges, this gives cycles of length $n2^{n-1}/2^{i_1+j-2+q} = n2^{n-i_1-j+1+q}$ for $0 \leq q \leq n-i_1-j$. Letting q vary from 0 to $n-i_1-j$ gives cycles of length $n2^m$ for all m from 1 (when $q = n-i_1-j$) to 2^{n-i_1-j+1} (when $q = 0$). □

6 Conclusions

In this paper, a method to decompose even hypercubes into cycles or paths of the same lengths is developed using special decompositions of toroidal graphs. It is shown, in particular, that Q_n decomposes into cycles of the same length that is as large as about $2^n/n$, which is a significant improvement over the previously known longest non-trivial lengths with odd divisors of n^2 in such a decomposition. Thus the main result of the paper makes a significant step towards resolving a conjecture of Erde.

The main Theorem 1 is supplemented with Proposition 12 that gives a different range of values for the cycle lengths. Tables 1 and 2 in the appendix give some examples of the cycle decompositions produced by these results. Note that even if we were just concerned with path decompositions of the hypercube, Theorem 1 gives some stronger results than Proposition 12. For example, the cycle decompositions of Q_{30} given by Proposition 12 has cycles of length at most $15 \cdot 2^{24}$ (in the notation of Proposition 12, $i_1 = 4$ and $j = 4$). Dividing these cycles in half gives paths with length $5(3 \cdot 2^{23})$. However as mentioned in the introduction, Theorem 1 gives cycles of length $5 \cdot 2^m$ for m as large as 27. Dividing these in half we get a path decomposition of Q_{30} into paths of length $5 \cdot 2^{26}$, and $2^{26} > 3 \cdot 2^{23}$. Proposition 12 gives more decompositions into short cycles in the case $x = 1$.

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References

[1] B. Alspach, J. Bermond, and D. Sotteau. Decomposition into cycles I: Hamilton decompositions. In G. H. et al. (editors) Cycles and rays. Kluwer Academic (1990), 9–18.

[2] D. Anick and M. Ramras, Edge decompositions of hypercubes by paths. Australas. J. Combin., 61 (2015), 210–226.

[3] J. Aubert and B. Schneider. Decomposition de la somme cartesienne d'un cycle et de l'union de deux cycles hamiltoniens en cycles hamiltoniens. Discrete Mathematics 38.1 (1982), 7–16.

[4] B. Auerbach, R. Laskar. On decomposition of r-partite graphs into edge-disjoint Hamilton circuits. Discrete Mathematics 14.3 (1976), 265–268.

[5] D. Bass and I. Sudborough. Hamilton Decompositions and $(n/2)$-Factorizations of Hypercubes. J. Graph Algorithms Appl., 7.1 (2003), 79–98.

[6] J. Bermond. Hamiltonian decompositions of graphs, directed graphs and hypergraphs. Advances in graph theory, Ann. Discrete Math. 3 (1978).

[7] M. Bonamy, N. Morrison, and A. Scott. Partitioning the vertices of a torus into isomorphic subgraphs, preprint arXiv:1710.07255 (2017).

[8] J. Erde. Decomposing the cube into paths. Discrete Math., 336 (2014), 41–45.

[9] J. Fink. On the decomposition of n-cubes into isomorphic trees. J. Graph Theory 14 (1990) 405–411.

[10] M. Foregger. Hamiltonian decompositions of products of cycles. Discrete Mathematics, 24.3 (1978), 251–260.

[11] S. Glock, D. Kühn, A. Lo, and D. Osthus. The existence of designs via iterative absorption, preprint arXiv:1611.06827 (2016).

[12] V. Gruslys. Decomposing the vertex set of a hypercube into isomorphic subgraphs, preprint arXiv:1611.02021 (2016).

[13] P. Horak, J. Siran, and W. Wallis. Decomposing Cubes, J. Austral. Math. Soc. Ser. A 61.1 (1996), 119–128.

[14] P. Keevash. The existence of designs, preprint arXiv:1401.3665 (2014).

[15] A. Kotzig. Every Cartesian product of two circuits is decomposable into two hamiltonian circuits, Centre de Recherches Mathematique, Montreal, (1973).

[16] D. Lucas. Récréations Mathématiques, Vol. II. Gauthier Villars, Paris, (1892).

[17] D. Offner, Packing the hypercube. Discussiones Mathematicae Graph Theory, 34.1 (2014), 85–93.

[18] G. Ringel. Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Vol. 20, No. 1-2 (1955).

[19] A. Roy and J. Kureethara. Decomposition of Hypercubes. International Journal of Pure and Applied Mathematics, 117.12 (2017), 265–272.

[20] R. Stong. Hamilton decompositions of Cartesian products of graphs. Discrete Mathematics 90.2 (1991), 169–190.

[21] Q. Stout. Packings in hypercubes, presented at 21st Southeastern Intl. Conf. on Combinatorics, Graph Theory, and Computing, Boca Raton, FL, (1990). Abstract available at http://web.eecs.umich.edu/~qstout/abs/hyppack.html.

[22] S. Tapadia, B. Waphare, and Y. Borse. Decompositions of n-cube into $2^m n$-cycles, preprint arXiv:1804.01243 (2018).

[23] R. Wilson. Decompositions of complete graphs into subgraphs isomorphic to a given graph, Proc. 5th British Combinatorial Conf, (1975).

A Numerical examples

n	α	x	y	i_1	j	$n-i_1-2xj$	Number of cycles	Cycle lengths
14	1	1	7	2	3	6	$\{2^q : 4 \le q \le 10\}$	$\{7 \cdot 2^m : 4 \le m \le 10\}$
14	1	7	1	0	1	0	$\{7 \cdot 2^q : 4 \le q \le 0\}$	$\{2^{14}\}$
30	1	1	15	3	4	19	$\{2^q : 6 \le q \le 25\}$	$\{15 \cdot 2^m : 5 \le m \le 24\}$
30	1	3	5	2	2	16	$\{3 \cdot 2^q : 3 \le q \le 22\}$	$\{5 \cdot 2^m : 11 \le m \le 27\}$
30	1	5	3	1	2	9	$\{5 \cdot 2^q : 2 \le q \le 11\}$	$\{3 \cdot 2^m : 19 \le m \le 28\}$
30	1	15	1	0	1	0	$\{15 \cdot 2^q : 0 \le q \le 0\}$	$\{2^{30}\}$
180	2	1	45	5	4	167	$\{2^q : 9 \le q \le 176\}$	$\{45 \cdot 2^m : 5 \le m \le 172\}$
180	2	3	15	3	4	153	$\{3 \cdot 2^q : 7 \le q \le 160\}$	$\{15 \cdot 2^m : 21 \le m \le 174\}$
180	2	9	5	2	2	142	$\{9 \cdot 2^q : 4 \le q \le 146\}$	$\{5 \cdot 2^m : 35 \le m \le 177\}$
180	2	5	9	3	2	157	$\{5 \cdot 2^q : 5 \le q \le 162\}$	$\{9 \cdot 2^m : 19 \le m \le 176\}$
180	2	15	3	1	2	119	$\{15 \cdot 2^q : 3 \le q \le 122\}$	$\{3 \cdot 2^m : 62 \le m \le 178\}$
180	2	45	1	0	1	90	$\{45 \cdot 2^q : 1 \le q \le 91\}$	$\{2^m : 90 \le m \le 180\}$

Table 1: The cycle lengths of the cycle decompositions of Q_n in Theorem 1.

n	i_1	j	$n-i_1-j$	Number of cycles	Cycle lengths
14	3	3	8	$\{2^q : 4 \le q \le 12\}$	$\{7 \cdot 2^m : 2 \le m \le 10\}$
30	4	4	22	$\{2^q : 6 \le q \le 28\}$	$\{15 \cdot 2^m : 2 \le m \le 24\}$
180	7	4	169	$\{2^q : 9 \le q \le 178\}$	$\{45 \cdot 2^m : 3 \le m \le 172\}$

Table 2: The cycle lengths of the cycle decompositions of Q_n in Proposition 12.