A translation of a paper by S. V. Sevastianov
"Interval colorability of the edges of a bipartite graph"

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For $a, b \in \{0, 1, 2, \ldots\}$, $a < b$, we denote by $(a, b]$ the interval $\{a+1, \ldots, b\}$. A function $\varphi\colon E \to (0, t]$ is called an *edge-coloring* of G in t colors, where $\varphi(e)$ is called the color of an edge e, for $e \in E$. An edge-coloring of G is *proper* if any two adjacent edges have distinct colors. For $x \in V$, let $\varphi(x) = \{\varphi(xy) : xy \in E\}$. A proper edge-coloring $\varphi\colon E \to (0, t]$ of G is an *interval coloring* in t colors if $\varphi^{-1}(1) \neq \emptyset$, $\varphi^{-1}(t) \neq \emptyset$, and the set $\varphi(x)$ is an interval of integers for any vertex x.

Note that for any interval coloring $\varphi\colon E \to (0, t]$ of a graph (V, E) in t colors, there is a symmetric interval coloring $\varphi'\colon E \to (0, t]$, where $\varphi'(e) = t + 1 - \varphi(e)$ for any edge e. The notion of interval colorings is introduced in [1].

For arbitrary graph G, it was shown in [1] that the problem of interval colorability is $\mathcal{NP}$-complete. In the case of bipartite graphs, Kamalian [3] identified two subclasses of interval-colorable graphs: complete bipartite graphs and trees. Moreover, any regular bipartite graph is interval colorable. That follows from the fact that any bipartite graph has a proper edge-coloring using colors $\{1, \ldots, \Delta\}$, where Δ is the maximum degree of the graph, see [5, ca. 388].

In each of these three cases, the interval coloring would be found in polynomial time. It remains to determine whether any bipartite graph is interval colorable. An example giving a negative answer to this question is given in Figure 1. In this paper we show that the problem of determining whether a bipartite graph is interval colorable is $\mathcal{NP}$-complete.

In [3] it was shown that trees and complete bipartite graphs satisfy the following property $(*)$: Let $w(G)$ and $W(G)$ be the minimum and maximum number of colors in an interval coloring of G. Then for any $t \in [w(G), W(G)]$, G is interval colorable in t colors.

It is easy to see that any regular interval colorable graph satisfies $(*)$. Indeed, if G is interval colorable in t colors, $(t > \Delta(G) = w(G))$, then by decreasing for each edge of color t its color by $\Delta(G)$, we shall get an interval coloring in $t - 1$ colors.

It is natural to ask whether any interval colorable bipartite graph satisfies $(*)$. At the end of the paper we give an example giving a negative answer to this question.

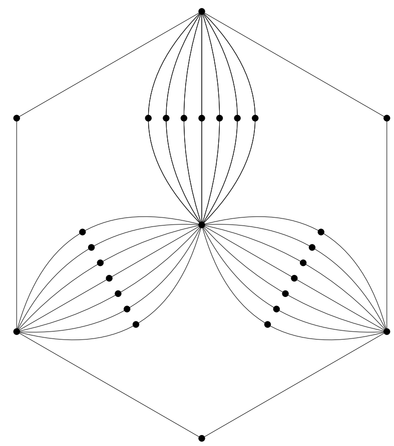

Figure 1

2 The $\mathcal{NP}$-Completeness of ICBG

Interval Coloring of Bipartite Graphs (ICBG)

Instance Simple bipartite graph $G = (V_1, V_2; E)$ and $t \in \mathbb{N}$.

Question Is there an interval coloring of G in t colors?

We shall prove the $\mathcal{NP}$-completeness of this problem via pseudo-polynomial reduction of a scheduling problem to ICBG. This scheduling problem described below is $\mathcal{NP}$-complete [2, 102–104]. The definition of length$[H]$ and max$[H]$ is given in [2, 92–95]. Roughly speaking, these functions correspond to the length of the input and the maximum of the absolute value of the input for a problem instance H, respectively.

Sequencing Within Intervals (SWI) [2, 70]

Instance Finite set of tasks $N = \{1, 2, \ldots, n\}$ and for each $i \in N, r_i, d_i, l_i \in \mathbb{Z}_+$ are the release time, deadline and length of i, respectively.

Question Is there a schedule for N, i.e. a function $\sigma\colon N \to \mathbb{Z}_+$, such that $\forall\, i \in N$
(1) $\sigma(i) \geq r_i$,
(2) $\sigma(i) + l_i \leq d_i$,
(3) if $j \in N \setminus \{i\}$, then either $\sigma(j) + l_j \leq \sigma(i)$ or $\sigma(j) \geq \sigma(i) + l_i$?

Let a set of intervals of integers be given as $\{I_i \colon i = 1, \ldots, m\}, I_i = (r_i, d_i] \subset \mathbb{Z}_+$. We shall construct a corresponding bipartite graph $G_{m,\{I_i\}}$, see Figure 2, where $l = m + 6; M = D + 2m + 10; D = D' + \delta(D'); D' = \max_i d_i$;

$$\delta(k) \coloneqq \begin{cases} 0, & k \text{ even}, \\ 1, & k \text{ odd}. \end{cases}$$

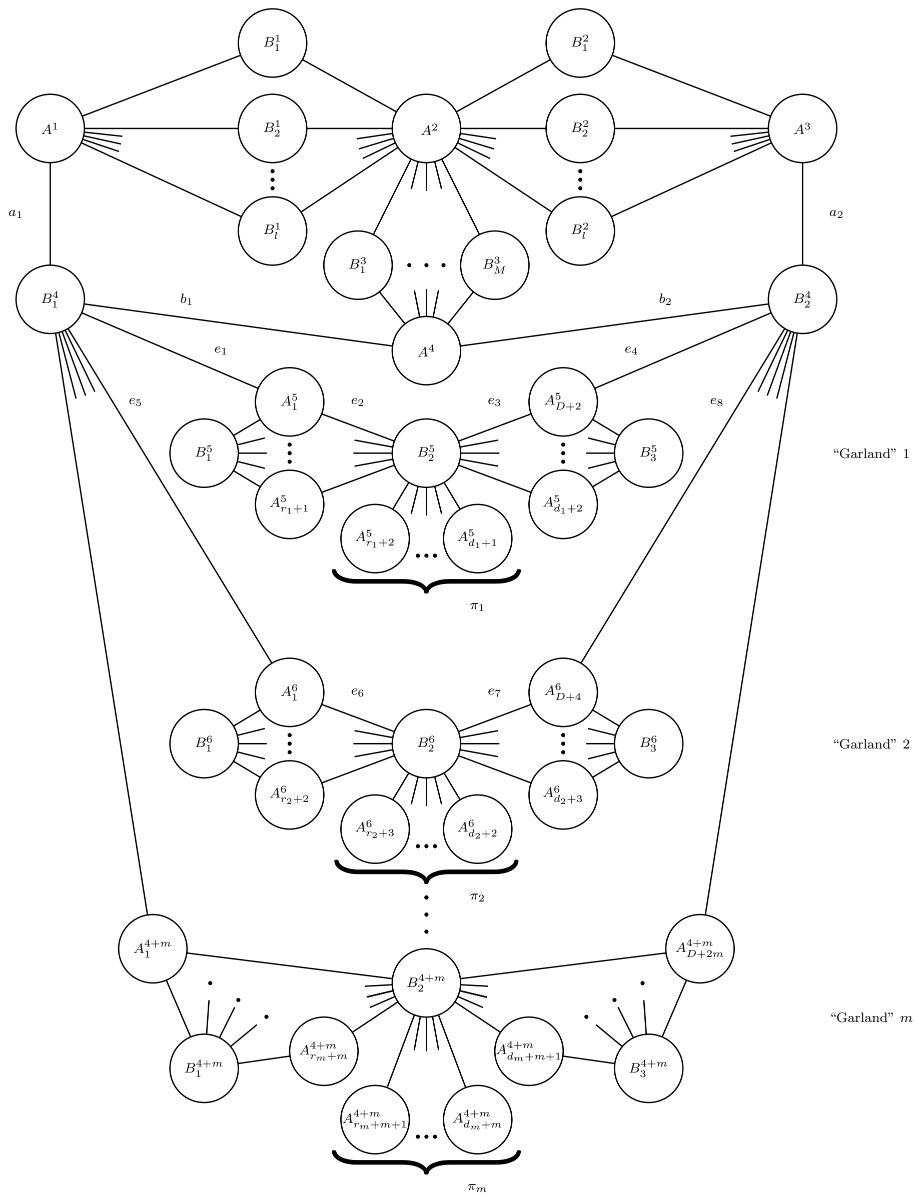

Figure 2: Graph $G_{m,\{I_i\}}$

The vertices of one part of $G_{m,\{I_i\}}$ are labeled A with subscripts and superscripts, the vertices of the other part are labeled B with subscripts and superscripts; π_i is a set of edges $\left\{B_2^{4+i}A_j^{4+i}\colon j = r_i + 1 + i, \ldots, d_i + i\right\}$, we call this set π_i an *outer bundle*.

Lemma 1. *$G_{m,\{I_i\}}$ is interval colorable. Moreover, for any interval coloring φ the following holds:*

a) the number of colors is $T = D + 4m + 22 = M + 2l$;

b) edges $a_1 = A^1B_1^4$ and $a_2 = A^3B_2^4$ have colors $\varphi(a_1) = c_0, \varphi(a_2) = T + 1 - c_0$ or $\varphi(a_1) = T + 1 - c_0, \varphi(a_2) = c_0$, where $c_0 = m + 8$;

c) $\varphi(\pi_i)$ is an interval for each i; moreover, if $\varphi(a_1) = c_0$, then $\varphi(\pi_i) = c_1 + (r_i, d_i]$ for any $i = 1, \dots, m$, where $c_1 = 2m + 11$.

Proof. Let φ be an arbitrary interval coloring of $G_{m,\{I_i\}}$. Let $\varphi_1 \coloneqq \varphi(A^1) \cup \varphi(A^3) \cup \varphi(A^4)$, $\nu_1 \coloneqq \min \varphi_1$, $\mu_1 \coloneqq \max \varphi_1$. Let $\nu_2 = \min \varphi(A^2), \mu_2 = \max \varphi(A^2)$. Since $\mu_2 - \nu_2 = M + 2l - 1, \nu_1 \leq \nu_2 + 1, \mu_1 \geq \mu_2 - 1$, then

$$\mu_1 - \nu_1 \geq M + 2l - 3. \tag{1}$$

From this, it is clear that ν_1 and μ_1 could not belong to the same set $\varphi(A^i)$, $i \in \{1, 3, 4\}$. If $\nu_1 \in \varphi(A^4), \mu_1 \in \varphi(A^3)$, then because of (1) there should be a gap of at least

$$(\mu_1 - \nu_1 + 1) - \left|\varphi(A^4)\right| - \left|\varphi(A^3)\right| \geq (M + 2l - 2) - (M + 2) - (l + 1) = l - 5 = m + 1$$

colors between $\varphi(A^3)$ and $\varphi(A^4)$. This can not happen since $\varphi(A^4) \cap \varphi(B_2^4) \neq \emptyset, \varphi(A^3) \cap \varphi(B_2^4) \neq \emptyset, \left|\varphi(B_2^4)\right| = m + 2$, from which it follows that the gap between $\varphi(A^4)$ and $\varphi(A^3)$ is at most m. Thus, either

$$\nu_1 \in \varphi(A^1) \text{ and } \mu_1 \in \varphi(A^3) \text{ or } \nu_1 \in \varphi(A^3) \text{ and } \mu_1 \in \varphi(A^1). \tag{2}$$

Let's assume w.l.o.g. that the first statement of (2) holds. From (2), we have $\varphi(a_1) - \nu_1 \leq l, \mu_1 - \varphi(a_2) \leq l$, and from (1) we see

$$\varphi(a_2) - \varphi(a_1) \geq \mu_1 - l - \nu_1 - l \geq M - 3. \tag{3}$$

Consider a path (e_1, e_2, e_3, e_4) connecting the edges a_1 and a_2 (see Figure 2). Since φ is interval, we have

$$\begin{aligned} |\varphi(a_1) - \varphi(a_2)| \leq{} & |\varphi(a_1) - \varphi(e_1)| + |\varphi(e_1) - \varphi(e_2)| + |\varphi(e_2) - \varphi(e_3)| \\ & + |\varphi(e_3) - \varphi(e_4)| + |\varphi(e_4) - \varphi(a_1)| \\ \leq{} & (m + 1) + 2 + (D + 1) + 2 + (m + 1) \\ ={} & D + 2m + 7 \\ ={} & M - 3. \end{aligned} \tag{4}$$

From (3) and (4), it follows that first, $\varphi(a_2) - \varphi(a_1) = M - 3$, second, that all intermediate inequalities in (1), (3), and (4) hold as equalities. Thus,

$$\begin{aligned}
\nu_2 &= 1,\\
\nu_1 &= 2,\\
\mu_2 &= M + 2l = D + 4m + 22 = T,\\
\mu_1 &= M + 2l - 1,\\
\varphi(a_1) &= l + 2 = m + 8,\\
\varphi(a_2) &= M + l - 1 = M + m + 5;\\
\varphi(e_1) &= \varphi(a_1) + m + 1 = 2m + 9,\\
\varphi(e_4) &= \varphi(a_2) - m - 1 = M + 4,
\end{aligned}$$

i.e. the color $\varphi(e_1)$ is maximal in the interval $\varphi(B_1^4)$, and $\varphi(e_4)$ is minimal in $\varphi(B_2^4)$;

$$\varphi(e_2) = \varphi(e_1) + 2 = 2m + 11, \varphi(e_3) = \varphi(e_4) - 2 = M + 2,$$

i.e. $\varphi(e_2)$ and $\varphi(e_3)$ are minimal and maximal in interval $\varphi(B_2^5)$;

$$\begin{aligned}
\varphi((A_1^5, B_1^5)) &= \varphi(e_2) - 1,\\
\varphi((A_{D+2}^5, B_3^5)) &= \varphi(e_3) + 1,\\
\varphi(B_1^5) &= \{2m + 10, \ldots, 2m + 10 + r_1\},\\
\varphi(B_3^5) &= \{2m + 13 + d_1, \ldots, M + 3\},\\
\varphi(\pi_1) &= \{2m + 12 + r_1, \ldots, 2m + 11 + d_1\} = c_1 + (r_1, d_1].
\end{aligned}$$

Considering the second "garland", that hangs on B_1^4, B_2^4 and contains the bundle π_2, note that $\varphi(e_5) = \varphi(e_1) - 1, \varphi(e_8) = \varphi(e_4) + 1, \varphi(e_8) - \varphi(e_5) = M + 4 - 2m - 9 + 2 = D + 2m + 10 - 2m - 3 = D + 7$, from which we have $\varphi(e_7) - \varphi(e_6) = D + 3$, i.e. again $\varphi(e_6)$ and $\varphi(e_7)$ are the minimum and maximum colors in $\varphi(B_2^6)$, from which we analogously obtain $\varphi(\pi_2) = c_1 + (r_2, d_2]$, and so on...

From the proven properties, it is not difficult to reconstruct the interval coloring of the whole graph. Since we basically gave an algorithm of coloring "garlands", it remains to explain how we color the central part of the graph. In Figure 4, it is shown how the colors from $\varphi(A^1), \varphi(A^4), \varphi(A^3)$ match the colors in $\varphi(A^2)$ at vertices B_j^i, $i \in \{1, 2, 3\}$.

A diagonal line connects two neighboring colors (one from $\varphi(A^2)$, another from one of the intervals $\varphi(A^1), \varphi(A^3), \varphi(A^4)$), that appear at some vertex B_j^i, $i \in \{1, 2, 3\}$. In addition, pairing up consecutive colors in

$$\{\varphi(b_1) + 1, \ldots, \varphi(b_2) - 1\} = \{m + 10, \ldots, M + m + 3\} \subset \varphi(A^4)$$

(see Figure 4) with the same colors from $\varphi(A^2)$ at B_j^3 is guaranteed to work due to the parity of the parameter D.

□

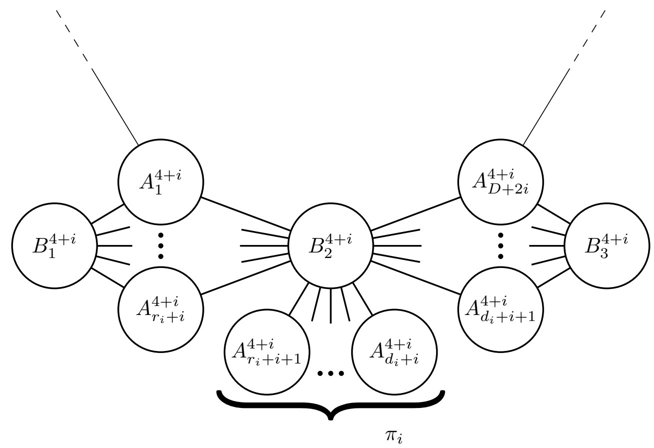

Figure 3: Anatomy of the i-th garland

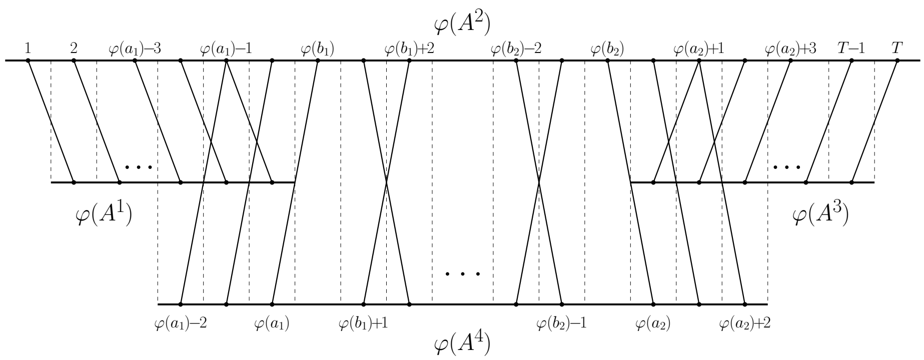

Figure 4

Theorem 2. *The problem* ICBG *is $\mathcal{NP}$-complete.*

Proof. Let us reduce the problem of SWI to ICBG. Let H be a specific instance of SWI with initial conditions $\{r_i, d_i, l_i : i = 1, \dots, n\} \subset \mathbb{Z}_+, \max[H] = D', \text{length}[H] \le M \log_2 D'$. Let us construct a bipartite graph $G(H)$ as shown in Figure 5.

We can assume that the following conditions are satisfied:

$$r_i + l_i \le d_i, i = 1, \dots, n, \tag{5}$$

$$L \doteq \sum_{i=1}^{n} l_i \le D'. \tag{6}$$

If some condition above is not satisfied, (that could be checked in linear of length$[H]$ time), then obviously there is no feasible schedule in problem H.

The graph $G(H)$ consists of the following parts:

- a graph $G^* = G_{2n,\{I_i, 2+I_i\colon i=1,\dots,n\}}$, where $I_i = (r_i, d_i]$,
- a graph $G^{**} = G_{2,\{(0,D'],4+(0,D']\}}$,
- vertices $\{v_i\colon i = 1,\dots,n\}$ with bundles of edges $\pi(v_i)$, $|\pi(v_i)| = l_i$,
- additional vertices $\left\{v_j^i\colon j = 1,\dots,d_i - r_i - l_i; i = 1,\dots,n\right\}$,
- additional vertices $\left\{\overline{v}_j^i\colon j = 1,2,3; i = L+1,\dots,D'\right\}$.

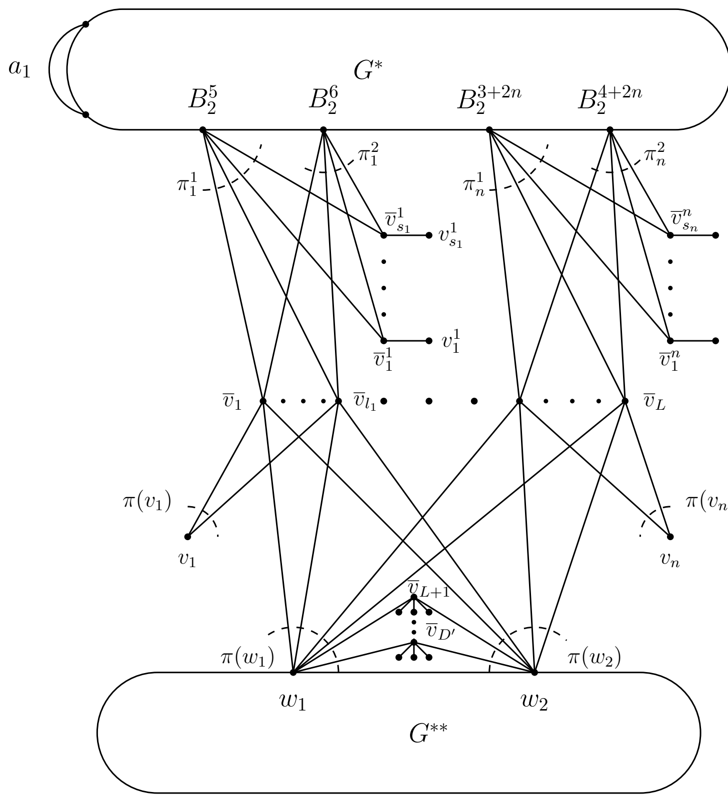

Figure 5: Graph $G(H)$

We allow for some additional vertices not to be included if inequalities in (5) or (6) for corresponding i hold as equalities. In G^*, the outer bundles π_i^1 and π_i^2 correspond to intervals I_i and $2 + I_i$. In G^{**} there are two outer bundles incident to vertices w_1 and w_2. Pendant vertices of the bundles $\left\{\pi_i^1, \pi_i^2, \pi(v_i), \pi(w_1), \pi(w_2)\right\}$ are connected as follows: In vertices $\overline{v}_i, i = 1,\dots,L$, the edges of the bundles $\left\{\pi_i^1, \pi_i^2, \pi(v_i), \pi(w_1), \pi(w_2)\right\}$ are connected.

If $s_i \doteq d_i - r_i - l_i > 0$, then the remaining s_i edges of the bundle π_i^1 and s_i edges from π_i^2 are pairwise connected at vertices $\overline{v}_j^i, j = 1,\dots,s_i$, adjacent to additional vertices v_j^i. The degree of each vertex $\overline{v}_j^i$ is 3.

If $D' - L > 0$, then the remaining $D' - L$ edges of the bundle $\pi(w_1)$ and $D' - L$ edges of the bundle $\pi(w_2)$ are pairwise connected at $\overline{v}_i, i = L+1, \ldots, D'$, adjacent also to additional vertices $\left\{\overline{v}^i_j : j = 1, 2, 3\right\}$.

The degree of each $\overline{v}_i, i = 1, \ldots, D'$ is 5; all additional vertices are pendant.

Next we shall show that a feasible schedule of H exists if and only if there is an interval coloring of $G(H)$. Let φ be an interval coloring of $G(H)$. Then by Lemma 1, G^* is colored in $T = D + 8n + 24$ colors from $(0, T]$.

We shall show that all other edges of $G(H)$ are colored in colors from this interval $(0, T]$: Assume that $\varphi(a_1) = 2n + 8$ (otherwise consider a symmetric coloring). Then by Lemma 1,

$$\varphi(\pi^1_i) = 4n + 11 + (r_i, d_i], \qquad \varphi(\pi^2_i) = 4n + 13 + (r_i, d_i].$$

From the lemma, we also know that $\varphi(\pi(w_1))$ and $\varphi(\pi(w_2))$ are two intervals, one obtained from another by shifting by 4. Let w_1 denote the vertex corresponding to the left interval, w_2 to the right interval. Pairwise adjacency of the edges of the bundles $\pi(w_1), \pi(w_2)$ appearing at $\overline{v}_j$, establishes a bijection between $\varphi(\pi(w_1))$ and $\varphi(\pi(w_2))$. Because $|\varphi((\overline{v}_j, w_1)) - \varphi((\overline{v}_j, w_2))| \leq 4$, $j = 1, \ldots, D'$, it is not difficult to see that for each j these are equalities, i.e. in the set of 5 colors $\varphi(\overline{v}_j)$ the colors $\varphi((\overline{v}_j, w_1)), \varphi((\overline{v}_j, w_2))$ are the minimum and maximum elements respectively.

From this, we have $\left|\varphi((\overline{v}_j, B_2^{3+2i})) - \varphi((\overline{v}_j, B_2^{4+2i}))\right| \leq 2$, and similar arguments give

$$\varphi((\overline{v}_j, B_2^{4+2i})) = \varphi((\overline{v}_j, B_2^{3+2i})) + 2.$$

Therefore, in each interval $\varphi(\overline{v}_j)$, the color $\varphi((\overline{v}_j, v_i))$ is in the center of the interval. From that we have

$$\varphi(v_i) \subset \varphi(\pi^1_i) + 1 = 4n + 12 + (r_i, d_i], i = 1, \ldots, n,$$

and the intervals $\{\varphi(v_i) : i = 1, \ldots, n\}$ do not intersect since the corresponding intervals of the edges of the bundle $\pi(w_1)$,

$$\varphi\left(\left\{(\overline{v}_j, w_1) : j = \sum_{k=1}^{i-1} l_k + 1, \ldots, \sum_{k=1}^{i} l_k\right\}\right) = \varphi(v_i) - 2, i = 1, \ldots, n,$$

do not intersect. (By the way, it allows us to establish that $\varphi(G^{**}) = (4n - 5, D + 4n + 29] \subset (0, D + 8n + 24] = \varphi(G^*)$ for any $n > 1$.)

From this, it is clear that assigning the i-th task to the interval $\varphi(v_i) - (4n + 12)$ gives us a feasible schedule for H.

Proving the other direction is not difficult since it is clear how given a schedule of H to get a coloring of $G(H)$. Since the number of edges of $G(H)$ is equal to

$$D \cdot (4n + 8) + D' + 8n^2 + 32n + 156 - 3L - \sum_{i=1}^{n} (d_i - r_i),$$

it does not exceed a polynomial of n and D' (and thus of length$[H]$ and max$[H]$).

Therefore, if there was an algorithm for ICBG that is polynomial on the number of edges of the graph, one would have gotten an in $\mathrm{length}[H], \max[H]$ polynomial algorithm for SWI, for any instance H. This contradicts the strong $\mathcal{NP}$-completeness of SWI (assuming $\mathcal{P} \neq \mathcal{NP}$). □

3 A Bipartite Graph Without Property (∗)

Next we shall construct a bipartite graph G, not satisfying (∗). Let $G_0 = G_{m,\{I_i\}}$ (as defined in Lemma 1) with $m = 1, D = D' = 74, I_1 = (r_1, d_1] = (0, 1]$.

According to Lemma 1, G_0 is interval colorable in $T = 100$ colors, and the color of the outer bundle consisting of a single edge is 14 (or in the symmetric coloring 87). Take G as the vertex disjoint union of two copies G', G'' of G_0 and then identify the outer edge of G' with the outer edge of G'' oriented in opposite direction, i.e. B_2^5 in one graph is identified with A_2^5 in another graph and vice-versa, see Figure 6. Then one can interval color G in 100 and 173 colors, but for any $t \in (100, 173)$, there is no interval coloring of G in t colors.

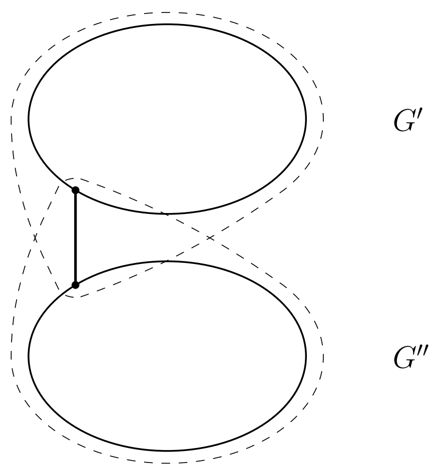

Figure 6: Graph G

4 Conclusion

The constructed graph in this paper and the main theorem show that properties of complete bipartite graphs and trees such as property (∗) found by Kamalian and the existence of a polynomial algorithm for interval coloring, do not hold for all bipartite graphs and thus are non-trivial properties for these graph classes.

Using this occasion, I would like to thank Ageev who noticed that the considered problem is related to the known problem 2-Dimensional Consecutive Sets, see [2, 230] under number [SR19]. It is equivalent to the interval colorability problem of an arbitrary hypergraph, whose $\mathcal{NP}$-completeness was shown in

[4] long before Asratian and Kamalian introduced the concept of interval coloring a graph.

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