Master thesis

Size-Ramsey Number of Uniform Hypergraph Paths

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Abstract

The size-Ramsey number $\hat{R}^{(k)}(\mathcal{H})$ of a k-uniform hypergraph $\mathcal{H}$ is the minimum number of edges such that there is a k-uniform hypergraph $\mathcal{G}$ on this many edges with the property that each edge 2-coloring of $\mathcal{G}$ contains a monochromatic copy of $\mathcal{H}$.

For $k \geq 2$, $1 \leq \ell \leq k-1$, $n \in \mathbb{N}$ with $\frac{n-\ell}{k-\ell} \in \mathbb{N}$, a k-uniform ℓ-path on n vertices $\mathcal{P}^{(k)}_{n,\ell}$ is defined as a k-uniform hypergraph on n vertices for which there is an ordering of its vertices such that every edge is a set of k consecutive vertices, two consecutive edges have precisely ℓ vertices in common, and there are no isolated vertices.

In this thesis the size-Ramsey number of k-uniform ℓ-paths $\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell})$ is examined in detail. Dudek, Fleur, Mubayi, and Rödl [12] showed the linearity of this number for the case $\ell \leq \frac{k}{2}$, while a linear upper bound for $\ell > \frac{k}{2}$ is a subject of current research. This work presents first estimates on a linear lower bound for this number for all choices of ℓ, with a particular focus on the loose case ($\ell = 1$) and the tight case ($\ell = k-1$). Most notably, it will be shown that for tight paths and for $k \geq 4$,

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,k-1}) \geq \left\lceil \log_2(k+1) \right\rceil \cdot n - 2k^2.$$

The thesis also provides an alternative approach for proving a lower bound in the graph case.

Contents

1 Introduction

1.1 Basic definitions

A graph in this thesis has the properties of being simple, undirected and finite. We follow the standard notation common in graph theory, see e.g. Diesel [11] and Bondi, Murty [7]. In particular, we use K_n and P_n for denoting *complete graphs* and *paths* on n vertices, respectively.

A *hypergraph* $\mathcal{G}$ is defined as a pair of a finite vertex set $V(\mathcal{G})$ and an edge set $E(\mathcal{G})$ consisting of non-empty subsets of $V(\mathcal{G})$. Similarly to the graph case, each hypergraph considered in this thesis is simple and undirected. If for a hypergraph $\mathcal{G}$ each edge in $E(\mathcal{G})$ is a k-set, i.e. a set on precisely k elements, then $\mathcal{G}$ is said to be *k-uniform* or equivalently a *k-graph*. This thesis entirely focusses on such uniform hypergraphs. It is easy to see that the definitions of 2-uniform hypergraphs and graphs are equivalent. As a consequence, the notation which is in the following introduced for uniform hypergraphs also apply to graphs. Conversely, most of the common definitions and notation for graphs can be directly transferred to hypergraphs, e.g. hypergraph isomorphisms, subhypergraphs and vertex degrees.

For $n \in \mathbb{N}$ let $[n] = \{1, \ldots, n\}$ and additionally it is convenient to define $[0] = \varnothing$. Given a set M and $k \in \mathbb{N}$, we define $M^{(k)}$ to be the set of all k-sets which are subsets of M. For two disjoint sets U, W we write $U \mathbin{\dot\cup} W$ for the union of these sets emphasizing the disjointness of U and W.

For a k-uniform hypergraph $\mathcal{G}$, we define $e(\mathcal{G}) = |E(\mathcal{G})|$. Furthermore, for an edge set Z, let $\bigcup Z = \{v \in e \colon e \in Z\}$ be the set of vertices that are *covered* by Z. Given a k-graph $\mathcal{G}$ and an edge set $Z \subseteq E(\mathcal{G})$ we say that Z *induces* the subhypergraph $(\bigcup Z, Z)$. For a vertex set $W \subseteq V(\mathcal{G})$, the subhypergraph of $\mathcal{G}$ *induced* by W is $(W, \{e \in E(\mathcal{G}) \colon e \subseteq W\})$. Observe that for $Z \subseteq E(\mathcal{G})$, the edge set Z and the vertex set $\bigcup Z$ possibly induce different subhypergraphs.

A *k-uniform complete hypergraph* on n vertices $\mathcal{K}_n^{(k)}$ is a k-graph isomorphic to $([n], [n]^{(k)})$, i.e. a k-graph containing all possible k-sets of its vertex set as edges. It can be seen that $\mathcal{K}_n^{(2)} = K_n$. Now let $n, k, \ell \in \mathbb{N}$ with $k \geq 2$, $1 \leq \ell \leq k-1$ and such that $\frac{n-\ell}{k-\ell} \in \mathbb{N}$. A *$k$-uniform ℓ-path* on n vertices $\mathcal{P}_{n,\ell}^{(k)}$ is a k-graph on n vertices for which there exists an ordering of its vertices such that every edge is a k-set of consecutive vertices (i.e. an interval of length k), two consecutive edges have precisely ℓ vertices in common, and every vertex has degree at least 1. More formally spoken, it is k-graph which is isomorphic to the k-graph $([n], E)$ where the edge set is

$$E = \left\{ \{(k-\ell)i + 1, (k-\ell)i + 2, \ldots, (k-\ell)i + k\} \colon i \in \{0, 1, \ldots, \tfrac{n-\ell}{k-\ell} - 1\} \right\}.$$

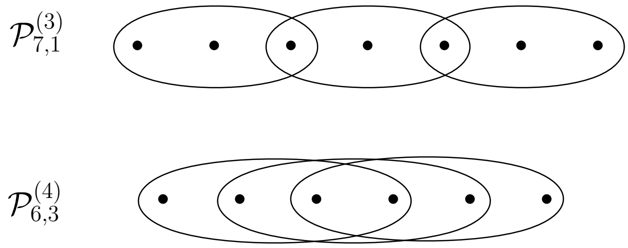

Figure 1: Hypergraph paths $\mathcal{P}^{(3)}_{7,1}$ and $\mathcal{P}^{(4)}_{6,3}$

Let $\mathcal{P}$ be a k-uniform ℓ-path on n vertices. The above described isomorphism implicitly provides an enumeration of both the vertices and the edges of $\mathcal{P}$. Such an enumeration is said to be a *path enumeration* of $V(\mathcal{P})$ or $E(\mathcal{P})$, respectively, regarding $\mathcal{P}$. Usually in this thesis we label the vertices as $V(\mathcal{P}) = \{v_1, \ldots, v_n\}$ and the edges as $E(\mathcal{P}) = \{e_1, \ldots, e_m\}$ where $m = e(\mathcal{P})$.

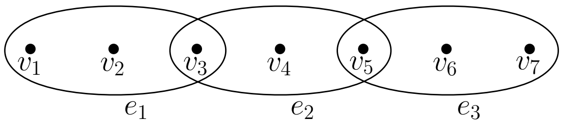

Figure 2: A path enumeration of the vertices and edges of a $\mathcal{P}^{(3)}_{7,1}$

It is clear to see that such an isomorphism is not unique, for instance consider the vertices in the reverse order, so there are multiple path enumerations of $\mathcal{P}$. In the following we will consider only at most one such labeling, which then "orients" $\mathcal{P}$. Two edges of the k-uniform ℓ-path $\mathcal{P}$ are *consecutive* if they intersect in precisely ℓ vertices. We say that an element of $E(\mathcal{P})$ is an *end edge* of $\mathcal{P}$ if there is only at most one edge consecutive to it. Note that every path edge which is not an end edge is consecutive to precisely two edges.

In addition to the enumeration of vertices and edges, we consider 2-sets of consecutive path edges, which we refer to as *segments* of $\mathcal{P}$. Using the above isomorphism there is also a *path enumeration* of the segments of a k-uniform ℓ-path. (For example, in the above Figure 2 the 2-set $\{e_1, e_2\}$ is the first segment and $\{e_2, e_3\}$ is the second segment.)

To simplify later formulas we introduce the function

$$m^{(k)}_{n,\ell} = e(\mathcal{P}^{(k)}_{n,\ell}) = \frac{n-\ell}{k-\ell}.$$

It should be highlighted that a k-uniform ℓ-path exists if and only if $m^{(k)}_{n,\ell} \in \mathbb{N}$, e.g. there does not exist a $\mathcal{P}^{(3)}_{6,1}$. We refer to k-uniform ℓ-paths as *loose* paths if $\ell = 1$ and as *tight* paths in the case $\ell = k-1$. Observe that a 2-uniform 1-path $\mathcal{P}^{(2)}_{n,1}$ on n vertices describes the same structure as a graph path P_n on n vertices. Also note that a k-uniform ℓ-path on at least two edges has maximum degree 2 if $\ell \leq \frac{k}{2}$.

Throughout this thesis, $k \in \mathbb{N}, k \geq 2$ denotes the fixed uniformity of the hypergraphs considered. If the uniformity is given from the context we omit the term 'k-uniform' in referring to ℓ-paths. When considering such structures, the variables n and ℓ are used as the further parameters of a hypergraph path $\mathcal{P}_{n,\ell}^{(k)}$, so for describing the number of involved vertices and the overlap of consecutive edges, respectively. In this context we always suppose that $\ell \in \mathbb{N}, 1 \leq \ell \leq k-1$ and $n \in \mathbb{N}$ with $m_{n,\ell}^{(k)} \in \mathbb{N}$. For most of the results in this thesis we focus on the consideration asymptotically in terms of n, so it is convenient to let n be chosen sufficiently large whenever necessary.

An *edge r-coloring* of a k-graph $\mathcal{G}$ is a function $c\colon E(\mathcal{G}) \to [r]$ that maps each edge to one of the given colors $1, \ldots, r$. In this thesis we exclusively restricting our view to edge 2-colorings. By that reason we omit the prefix and refer to such a function simply as a *coloring* of $\mathcal{G}$. Moreover in order to improve the readability, we replace the used colors by *red* and *blue*. Occasionally, we define colorings of a k-graph $\mathcal{G}$ by firstly constructing a *partial coloring* of $\mathcal{G}$, which is a partial function $c\colon E(\mathcal{G}) \dashrightarrow \{\text{red}, \text{blue}\}$, and assigning colors to the remaining uncolored edges at a later time.

We say that a k-graph $\mathcal{G}$ has the *Ramsey property* $\mathcal{G} \to \mathcal{H}$ for some k-graph $\mathcal{H}$ if each coloring of $\mathcal{G}$ contains a monochromatic copy of $\mathcal{H}$.

1.2 History of Ramsey numbers

The combinatorical field of Ramsey theory can be traced back to a publication of F.P. Ramsey [28] in 1930. He proved that for each $n \in \mathbb{N}$, $r \geq 2$, $k \geq 2$ there is a natural number $N \in \mathbb{N}$ such that each edge r-coloring of $\mathcal{K}_N^{(k)}$ contains a monochromatic $\mathcal{K}_n^{(k)}$. His result raises the question which the minimal such value N is. Particularly, the special case $r = 2$, $k = 2$ leads to the following definition.

Definition 1.1. *The Ramsey number of a graph H is defined as*

$$R(H) = \min\Big\{N \in \mathbb{N}\colon K_N \to H\Big\}.$$

Note that Ramsey's theorem implies the existence of $R(H)$ for all H as $H \subseteq K_{|V(H)|}$.

Based on Ramsey's result Erdős and Szekeres [19] in 1935 and Erdős [15] in 1947 brought the topic to a broader audience by proving the following upper and lower bound on the Ramsey number of complete graphs,

$$2^{n/2} \leq R(K_n) \leq 2^{2n}.$$

The mentioned authors were the first to introduce the Ramsey number as a function depending on the considered graph, which is today's standard. Despite major efforts over the last 70 years, there was only a marginal improvement on the above bounds, see e.g. [30], [9], [29].

In the following years the research on Ramsey numbers was also extended beyond complete graphs, so that bounds on $R(H)$ were shown for many graph structures and classes of graphs for H, like paths [21], cycles [20] or graphs of bounded maximum degree [8]. For a general survey on current studies see [10].

While the classic Ramsey number considers the minimum number of vertices such that a complete host graph on this many vertices has the Ramsey property for some graph H, there is a different sense in which the host graph can be considered minimal. The size-Ramsey number, which was firstly introduced by Erdős, Faudree, Rousseau and Schelp [17] in 1978, is determined considering the minimum amount of edges in a host graph such that it has the Ramsey property.

Definition 1.2. *We denote the size-Ramsey number of a graph H by*

$$\hat{R}(H) = \min\Big\{N' \in \mathbb{N} \colon \exists \text{ graph } G \text{ with } e(G) = N', G \to H\Big\}.$$

Observe that Ramsey's theorem implies the existence of $\hat{R}(H)$ for all graphs H as $\hat{R}(H) \leq \binom{R(H)}{2} \leq \binom{R(K_n)}{2}$ for $n = |V(H)|$. An overview on current research concerning size-Ramsey numbers is given in [10].

A focus point of studies on this number is estimating the size-Ramsey number of paths. In 1981 Erdős [16] offered a prize of 100$ for a proof or disproof of the claim that $\hat{R}(P_n) = \omega(n)$ and $\hat{R}(P_n) = o(n^2)$. The question was answered negatively by Beck [4], who showed that $\hat{R}(P_n) < 900n$ for sufficiently large n. This upper bound has been improved several times in [6], [13], [23] with the state of the art result being $\hat{R}(P_n) \leq 74n$ given by Dudek and Pralat [14] (see Theorem 2.1). In Subsection 2.1 of this thesis the known upper bounds of the size-Ramsey number of paths in the graph case are considered in more detail.

Furthermore, the lower bound of this size-Ramsey number is considered in several publications as well, e.g. [5], [6], [14]. The best proven bound is $\hat{R}(P_n) \geq \Big(\frac{15}{4} - o(1)\Big)n$ given by Bal and DeBiasio [3]. For a more in-depth discussion on this lower bound the reader is referred to Subsection 2.2.

Notably, Ramsey's theorem applies not only to graphs but also to k-uniform hypergraphs. Consequently, it is an obvious step to generalize Ramsey-type problems to hypergraphs.

Definition 1.3. *The Ramsey number of a k-graph $\mathcal{H}$ is*

$$R^{(k)}(\mathcal{H}) = \min\Big\{N \in \mathbb{N} \colon \mathcal{K}_N^{(k)} \to \mathcal{H}\Big\}.$$

The hypergraph case of the classic Ramsey problem was first detailed examined by Erdős, Hajnal and Rado [18] in 1965. For a current survey on Ramsey-type problems for uniform hypergraphs the reader is referred to [27]. Note that there are also studies discussing Ramsey numbers of hypergraphs that does not have a certain uniformity, e.g. Berge hypergraphs, see [2].

In this thesis the main focus is placed on size-Ramsey problems for uniform hypergraphs.

Definition 1.4. *The size-Ramsey number of a k-uniform hypergraph $\mathcal{H}$ is defined as*

$$\hat{R}^{(k)}(\mathcal{H}) = \min\Big\{N' \in \mathbb{N} \colon \exists \text{ } k\text{-graph } \mathcal{G} \text{ with } e(\mathcal{G}) = N', \mathcal{G} \to \mathcal{H}\Big\}.$$

Note that existence of these numbers is once more guaranteed by Ramsey's theorem. The research of questions related to the size-Ramsey number of uniform hypergraphs was substantially driven forward by a paper of Dudek, La Fleur, Mubayi and Rödl [12] in 2015. Besides considering complete hypergraphs and other hypergraph structures, they gave first bounds on the size-Ramsey number of k-uniform ℓ-paths. With a view to this thesis it should be highlighted that the mentioned authors proved the linearity of this number in the case $1 \le \ell \le \frac{k}{2}$ by showing that

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \le \hat{R}(P_n), \qquad \text{for } 1 \le \ell \le \tfrac{k}{2}.$$

For $\ell > \frac{k}{2}$, their publication did not provide a linear upper bound for the size-Ramsey number of k-uniform ℓ-paths, although they conjectured the existence of such a bound (see Conjecture 2.6). In the case $k = 3$ this conjecture has been proven by Han et al. [22] who showed that

$$\hat{R}^{(3)}(\mathcal{P}^{(3)}_{n,2}) \le O(n).$$

For further publications considering the size-Ramsey number of hypergraphs, see [24] and [26]. The known bounds on the size-Ramsey number of k-uniform ℓ-paths are further discussed in Subsection 2.1.

1.3 Results of this thesis

This thesis is building on the survey by Dudek, La Fleur, Mubayi and Rödl [12] with a focus on the size-Ramsey number of k-uniform ℓ-paths. The main subject of this work is to give first lower bounds on that number. Most relevantly, loose and tight paths are considered. For the former we show the following bounds.

Theorem 1.5. *Let $3 \le k \le 7$ and let $n \in \mathbb{N}$ sufficiently large with $m^{(k)}_{n,1} \in \mathbb{N}$, then*

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,1}) \ge \left(2 + \frac{1}{2k-2}\right) m^{(k)}_{n,1} - 4.$$

Theorem 1.6. *Let $k \geq 8$ and let $n \in \mathbb{N}$ sufficiently large with $m^{(k)}_{n,1} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,1}) > \left(2 + \frac{1}{13}\right) m^{(k)}_{n,1} - 6, \qquad \text{for } k = 8,$$

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,1}) > \left(2 + \frac{1}{14}\right) m^{(k)}_{n,1} - 6, \qquad \text{for } 9 \leq k \leq 10 \text{ and}$$

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,1}) > \left(2 + \frac{1}{k+3}\right) m^{(k)}_{n,1} - 6, \qquad \text{for } k \geq 11.$$

Regarding tight paths, this thesis provides substantially stronger lower bounds.

Theorem 1.7. *Let $n \in \mathbb{N}$ sufficiently large. Then*

$$\hat{R}^{(3)}(\mathcal{P}^{(3)}_{n,2}) \geq \tfrac{8}{3} m^{(3)}_{n,2} - 4 = \tfrac{8}{3} n - \tfrac{28}{3}.$$

Theorem 1.8. *Let $k \geq 4$ and let $n \in \mathbb{N}$ sufficiently large. Then*

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,k-1}) \geq \left\lceil \log_2(k+1) \right\rceil \cdot m^{(k)}_{n,k-1} - k^2 \geq \left\lceil \log_2(k+1) \right\rceil \cdot n - 2k^2.$$

This theorem especially implies a result on the asymptotic behavior of the size-Ramsey number of tight paths, which is

$$\liminf_{k \to \infty} \liminf_{n \to \infty} \frac{\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,k-1})}{n} = \infty.$$

In addition, we also consider k-uniform ℓ-paths for $1 < \ell < k-1$. On the one hand, there is a generalization of the bound to tight paths stated in Theorem 1.8.

Theorem 1.9. *Let $k \geq 4$ and $\frac{2}{3}k < \ell \leq k-1$. Let $n \in \mathbb{N}$ sufficiently large with $m^{(k)}_{n,\ell} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \geq \left\lceil \log_2\left(\frac{2k-\ell}{k-\ell}\right) \right\rceil \cdot m^{(k)}_{n,\ell} - 4k^2.$$

For the remaining unconsidered values of ℓ we prove the following estimate.

Theorem 1.10. *Let $k \geq 2$, $1 \leq \ell \leq \frac{2}{3}k$. Let $n \in \mathbb{N}$ sufficiently large such that $m^{(k)}_{n,\ell} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \geq \left(2 + \frac{\ell}{2k-\ell-1}\right) \cdot m^{(k)}_{n,\ell} - 4.$$

Furthermore,

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \geq \tfrac{5}{2} m^{(k)}_{n,\ell} - 4, \qquad \text{if } \ell = \tfrac{k}{2}, \text{ and}$$

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \geq \tfrac{8}{3} m^{(k)}_{n,\ell} - 4, \qquad \text{if } \ell = \tfrac{2}{3}k.$$

This result directly implies Theorem 1.5 and Theorem 1.7. Additionally, it is noteworthy that Theorem 1.10 also provides a lower bound for the graph case (so for uniformity $k = 2$).

Corollary 1.11. *For $n \in \mathbb{N}$ sufficiently large, it holds that*

$$\hat{R}(P_n) \geq \tfrac{5}{2}n - \tfrac{15}{2}.$$

This estimate is not an improvement of the current best known bound, but yields an alternative proof which differs from the known approaches.

Section 2 of this thesis summarizes the known upper and lower bounds in the graph and hypergraph case with regard to this thesis. Worth mentioning is Proposition 2.3 which yields an improvement of the linear upper bound of Dudek et al. [12]. In fact we show for $1 \leq \ell \leq \frac{k}{2}$ that

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \leq \hat{R}(P_{m^{(k)}_{n,\ell}+1}) \leq 74m^{(k)}_{n,\ell} + 74,$$

a result which follows from going prudently through the original proof of Dudek et al. Afterwards, Section 3 discusses the generalization of the known proof concepts which are used for lower bounds in the graph case to k-uniform hypergraphs.

The main results stated above are shown using two different approaches. Section 4 of this thesis focusses on the first of them, the *crossing approach*. After introducing the notation in Subsection 4.1, it is presented in Subsection 4.2 how to use this approach to obtain a bound for ℓ-paths with $\ell \leq \frac{k}{2}$. As a refinement, Theorem 1.6 will be proven in Subsection 4.3.

The second proof technique, which is referred to as the *neighborhood approach*, is introduced in Subsection 5.1. Subsection 5.2 yields a proof for Theorem 1.8, while in Subsection 5.3 the proofs of Theorems 1.9 and 1.10 are presented.

2 Graph Case and Basic Conditions

2.1 Known upper bounds

Following Beck's result $\hat{R}(P_n) < 900n$ for sufficiently large n in [4] which first verifies the linearity of the size-Ramsey number of paths in the graph case, the refinement of this upper bound has been considered in several publications. Gradually, the best known bound was improved with Bollobás [6] showing that $\hat{R}(P_n) < 720n$, followed by Dudek and Pralat [13] proving $\hat{R}(P_n) < 137n$. Letzter [23] showed that $\hat{R}(P_n) \leq 91n$ and finally the current best known bound is the following result due to Dudek and Pralat [14].

Theorem 2.1. *[14] Let $n \in \mathbb{N}$ sufficiently large. Then*

$$\hat{R}(P_n) \leq 74n.$$

The proofs in all above mentioned publications are using probabilistic methods. Thereby the existence of a graph G on few edges with the Ramsey property $G \to P_n$ is shown by verifying that with a probability strictly larger than 0 an arbitrary random graph (drawn from various random graph models) is as required. Especially, this proof technique is non-constructive, so does not provide such a graph explicitly. However, there is a constructive proof of the linearity given by Alon and Chung [1] building on a construction by Lubotzky et al. [25].

Theorem 2.2. *[1] Let $n \in \mathbb{N}$ sufficiently large. Then there is an explicitly constructable graph G such that*

$$G \to P_n \quad \text{and} \quad e(G) = O(n).$$

For k-uniform ℓ-paths, the research for an upper bound was initiated by Dudek, La Fleur, Mubayi and Rödl in [12]. Among other results, they stated that for $n \in \mathbb{N}, k \geq 2$ and $1 \leq \ell \leq \frac{k}{2}$ with $m_{n,\ell}^{(k)} \in \mathbb{N}$,

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \leq \hat{R}(P_n),$$

so that an upper bound on the size-Ramsey number of graph paths implies a comparable bound for ℓ-paths if $\ell \leq \frac{k}{2}$. In the following we show a refinement of this result.

Proposition 2.3. *Let $k \geq 3$ and $1 \leq \ell \leq \frac{k}{2}$. Let $n \in \mathbb{N}$ sufficiently large with $m_{n,\ell}^{(k)} \in \mathbb{N}$ and $n' = m_{n,\ell}^{(k)} + 1$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \leq \hat{R}(P_{n'}) \leq 74m_{n,\ell}^{(k)} + 74,$$

Proof. The second inequality is immediate by Theorem 2.1. In order to show the first inequality, let $r = \hat{R}(P_{n'})$ and fix a graph G with $e(G) = r$ and $G \to P_{n'}$.

We construct a k-graph $\mathcal{H}$ with $e(\mathcal{H}) = e(G)$ and $\mathcal{H} \to \mathcal{P}^{(k)}_{n,\ell}$ by "blowing up" G, which means that for each edge in the graph G we define a corresponding k-set to be an edge in the k-graph $\mathcal{H}$.

Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_r\}$ be arbitrarily enumerated. For each vertex v_i, $i \in [n]$, we define ℓ many vertices $v_i^1, \ldots, v_i^\ell$ and for each edge e_i, $i \in [r]$, let $w_i^1, \ldots, w_i^{k-2\ell}$ be further vertices such that all mentioned vertices are pairwise distinct. Note that if $\ell = \frac{k}{2}$ no vertices w_i^j are defined. The vertex set of $\mathcal{H}$ is the union of those vertices,

$$V(\mathcal{H}) = \left\{v_i^j : i \in [n], j \in [\ell]\right\} \cup \left\{w_i^j : i \in [r], j \in [k-2\ell]\right\}.$$

For each edge $e_i = \{v_{i_1}, v_{i_2}\} \in E(G)$, let

$$e_i' = \{v_{i_1}^1, \ldots, v_{i_1}^\ell\} \cup \{v_{i_2}^1, \ldots, v_{i_2}^\ell\} \cup \{w_i^1, \ldots, w_i^{k-2\ell}\}.$$

Then let the edge set of $\mathcal{H}$ be

$$E(\mathcal{H}) = \left\{e_i' : i \in [r]\right\}.$$

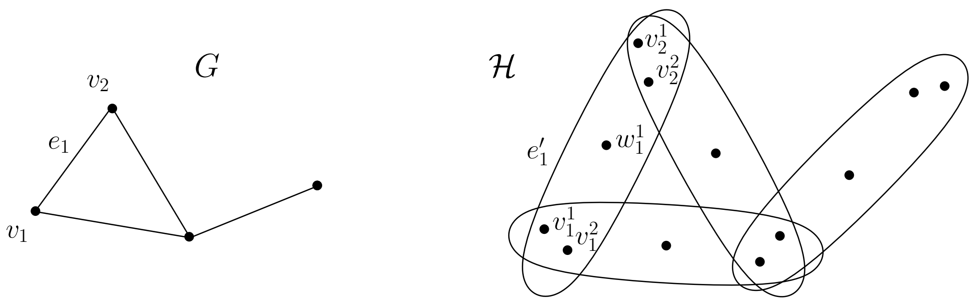

Figure 3: Construction of $\mathcal{H}$ for $k = 5$, $\ell = 2$

Clearly, $\mathcal{H}$ is a k-uniform hypergraph and $e(G) = e(\mathcal{H})$, so showing that $\mathcal{H} \to \mathcal{P}^{(k)}_{n,\ell}$ completes the proof. Pick an arbitrary edge 2-coloring of $\mathcal{H}$. We shall prove that there is a monochromatic $\mathcal{P}^{(k)}_{n,\ell}$ in this coloring. Using the canonical bijection between $E(\mathcal{H})$ and $E(G)$, $e_i' \mapsto e_i$ for all $i \in [r]$, we obtain a coloring of G. Since $G \to P_{n'}$, there is a monochromatic $P_{n'}$ in G, which we denote by P.

Obviously, $e(P) = n' - 1$, so without loss of generality, let $E(P) = \{e_1, \ldots, e_{n'-1}\}$. Then it can be seen that $\{e_1', \ldots, e_{n'-1}'\}$ is the edge set of a k-uniform ℓ-path in $\mathcal{H}$. Using the fact that $n' - 1 = m^{(k)}_{n,\ell}$, this ℓ-path consists of precisely n vertices. Since P is monochromatic in G, the ℓ-path is monochromatic in $\mathcal{H}$ and thus as required. $\square$

In view of this thesis it is worth highlighting that this result provides an upper bound of the form $c \cdot m^{(k)}_{n,\ell}$ where the leading factor $c > 0$ is constant in terms of n and k. Analogously to the above proof, Theorem 2.2 is transferable to ℓ-paths for $\ell \leq \frac{k}{2}$.

Proposition 2.4. *Let $k \geq 3$, $1 \leq \ell \leq \frac{k}{2}$. Let $n \in \mathbb{N}$ sufficiently large such that $m_{n,\ell}^{(k)} \in \mathbb{N}$. Then there exists an explicitly constructable k-graph $\mathcal{G}$ with*

$$\mathcal{G} \to \mathcal{P}_{n,\ell}^{(k)} \quad \text{and} \quad e(\mathcal{G}) = O(n).$$

□

For $\ell > \frac{k}{2}$, it is not possible to apply such an argument since in this case an ℓ-path contains vertices of degree 3. It is a topic of current research to determine an as good as possible upper bound for those values of ℓ. Concerning this, the consideration of tight paths, i.e. the case $\ell = k-1$, is especially interesting because an upper bound for $\ell = k-1$ also implies a bound for general ℓ as the following observation shows.

Observation 2.5. *Let $k \geq 2$ and $1 \leq \ell \leq k-1$. Let $n \in \mathbb{N}$ with $m_{n,\ell}^{(k)} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \leq \hat{R}^{(k)}(\mathcal{P}_{n,k-1}^{(k)}).$$

Proof. It is easy to see that $\mathcal{P}_{n,\ell}^{(k)} \subseteq \mathcal{P}_{n,k-1}^{(k)}$. Let $\mathcal{G}$ be a k-graph with $e(\mathcal{G}) = \hat{R}^{(k)}(\mathcal{P}_{n,k-1}^{(k)})$ and $\mathcal{G} \to \mathcal{P}_{n,k-1}^{(k)}$. Then each edge 2-coloring of $\mathcal{G}$ contains a monochromatic $\mathcal{P}_{n,k-1}^{(k)}$ and by this also contains a monochromatic $\mathcal{P}_{n,\ell}^{(k)}$. Consequently, $\mathcal{G} \to \mathcal{P}_{n,\ell}^{(k)}$. □

On the basis of the known bounds for graph paths and k-uniform ℓ-paths with $\ell \leq \frac{k}{2}$, Dudek et al. [12] stated the following conjecture.

Conjecture 2.6. *[12] Let $n \in \mathbb{N}, k \geq 3$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,k-1}^{(k)}) = O(n).$$

Combined with Observation 2.5, this conjecture implies that the size-Ramsey number of k-uniform ℓ-paths is linear in terms of n for all $1 \leq \ell \leq k-1$. Observe that Conjecture 2.6 is equivalent to the statement $\hat{R}^{(k)}(\mathcal{P}_{n,k-1}^{(k)}) \leq c \cdot m_{n,k-1}^{(k)}$ for some constant $c > 0$ independent from n (using the fact that $m_{n,k-1}^{(k)} = n-k+1$). The conjecture was partially verified by Han et al. [22] who answered the special case $k = 3$.

Theorem 2.7. *[22] Let $n \in \mathbb{N}$. Then*

$$\hat{R}^{(3)}(\mathcal{P}_{n,2}^{(3)}) = O(n).$$

In its general form the current best progress on Conjecture 2.6 is given by Lu and Wang [24] improving a result of Dudek et al. [12].

Theorem 2.8. *[24] Let $n \in \mathbb{N}$ and $k \geq 3$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,k-1}^{(k)}) = O\Big((n \log n)^{\frac{k}{2}}\Big).$$

2.2 Classification of the lower bound problem

Similarly to the previous subsection, we at first consider the lower bound for the size-Ramsey number of paths in the graph case. By applying the trivial lower bound (stated in detail in Observation 2.9 for $k = 2$) and the upper bound of Dudek and Pralat, Theorem 2.1, we obtain for sufficiently large $n \in \mathbb{N}$ the framework

$$2n - 3 \leq \hat{R}(P_n) \leq 74n.$$

The obvious key interest in refining the lower bound on this number is to determine a leading factor of n as good as possible.

The trivial bound was gradually improved in several publications. The basic concept for such proofs is to construct a coloring disproving the Ramsey property $G \to P_n$ for each graph G with few edges. In order to obtain this coloring, various different tools are used. Beck [5] proved $\hat{R}(P_n) \geq \left(\frac{9}{4} - o(1)\right)n$ by constructing colorings using a vertex partition based on vertex degrees. Bollobás [6] showed $\hat{R}(P_n) \geq \left(1 + \sqrt{2} - o(1)\right)n$ in an approach focusing on properties of spanning trees, a proof which was later refined by Dudek and Pralat [14] proving $\hat{R}(P_n) \geq \frac{5}{2}n - \frac{15}{2}$. Recently the best known bound leaped to $\hat{R}(P_n) \geq \left(\frac{15}{4} - o(1)\right)n$ due to a result given by Bal and DeBiasio [3]. Their proof combines several different approaches (including minimum degree conditions and spanning trees) in order to find a vertex partition with few crossing edges.

In contrast to graph paths there is no known lower bound in the case of k-uniform ℓ-paths, $k \geq 3$, except for the trivial one.

Observation 2.9. *Let $n \in \mathbb{N}$, $k \geq 2$, $1 \leq \ell \leq k-1$ with $m_{n,\ell}^{(k)} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \geq 2m_{n,\ell}^{(k)} - 1.$$

Proof. Let $\mathcal{G}$ be a k-graph with $e(\mathcal{G}) = 2m_{n,\ell}^{(k)} - 2$. Pick arbitrarily $m_{n,\ell}^{(k)} - 1$ many distinct edges and color them red. Color the remaining $m_{n,\ell}^{(k)} - 1$ many edges in blue. Clearly, this coloring does not contain a monochromatic $\mathcal{P}_{n,\ell}^{(k)}$ since such a subhypergraph consists of $m_{n,\ell}^{(k)}$ edges of the same color.

Consequently, $\mathcal{G} \nrightarrow \mathcal{P}_{n,\ell}^{(k)}$ for each $\mathcal{G}$ with $e(\mathcal{G}) = 2m_{n,\ell}^{(k)} - 2$. It is easy to see that each k-graph on at most $2m_{n,\ell}^{(k)} - 2$ edges is the subhypergraph of one such $\mathcal{G}$. Thus a k-graph with the Ramsey property for $\mathcal{P}_{n,\ell}^{(k)}$ consists of at least $2m_{n,\ell}^{(k)} - 1$ edges. □

If the parameter ℓ is restricted to values between $1 \leq \ell \leq \frac{k}{2}$, the approximate order of magnitude of the size-Ramsey number of k-uniform ℓ-paths is similar to the graph case. Combining Observation 2.9 and Proposition 2.3, we obtain

$$2m_{n,\ell}^{(k)} - 1 \leq \hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \leq 74m_{n,\ell}^{(k)} + 74, \quad \text{for } 1 \leq \ell \leq \tfrac{k}{2}.$$

So, similarly to the graph case, our main interest for a lower bound on $\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)})$ for $1 \leq \ell \leq \frac{k}{2}$ concerns the leading factor of $m_{n,\ell}^{(k)}$.

If $\ell > \frac{k}{2}$ and especially in the tight case $\ell = k-1$, the framework conditions are significantly wider. As presented in the previous subsection, the linearity of the size-Ramsey number $\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)})$ is only proven for $k=3$ and conjectured for $k \geq 4$ (see Theorem 2.7 and Conjecture 2.6, respectively). Clearly, a superlinear lower bound disproves the conjecture. Such a result is not achieved in this thesis, instead we as well focus on finding a leading factor of $m_{n,\ell}^{(k)}$ as large as possible.

It is worth mentioning that there is a connection between the lower bound problem for k-uniform ℓ-paths and graph paths obtained by reversely applying Proposition 2.3, i.e. using a lower bound in the k-uniform case for $k \geq 3$ and $1 \leq \ell \leq \frac{k}{2}$ to deduce a bound for the graph case $k=2$.

Observation 2.10. *Let $k \geq 3$, $\ell \leq \frac{k}{2}$. Let $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ be constants and suppose that for all $n \in \mathbb{N}$ sufficiently large with $m_{n,\ell}^{(k)} \in \mathbb{N}$,*

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \geq c_1 m_{n,\ell}^{(k)} - c_2.$$

Then for n sufficiently large

$$\hat{R}(P_n) \geq c_1 n - (c_1 + c_2).$$

Proof. Let $n_0 \in \mathbb{N}$ such that for all $N \geq n_0$ with $m_{N,\ell}^{(k)} \in \mathbb{N}$ it holds that

$$\hat{R}^{(k)}(\mathcal{P}_{N,\ell}^{(k)}) \geq c_1 m_{N,\ell}^{(k)} - c_2.$$

Consider $n \in \mathbb{N}$ with $n \geq \frac{n_0}{k-\ell}$ and let $n' = (k-\ell)(n-1)+\ell$. Note that $m_{n',\ell}^{(k)} = n-1$ and $n' \geq n_0$. Then applying Proposition 2.3

$$\hat{R}(P_n) \geq \hat{R}^{(k)}(\mathcal{P}_{n',\ell}^{(k)}) \geq c_1 m_{n',\ell}^{(k)} - c_2 \geq c_1 n - (c_1 + c_2).$$

□

In this sense, a lower bound on the leading factor c_1 is a stronger statement when considering ℓ-paths with $\ell \leq \frac{k}{2}$ than a bound with the same leading factor in the graph case. For $\ell > \frac{k}{2}$ a comparable observation is not known. (Note that such a result answers Conjecture 2.6.)

3 Generalization of known approaches

In order to find a lower bound on the size-Ramsey number of k-uniform ℓ-paths, it seems reasonable to firstly consider the approaches used for the known bounds in the graph case and attempt to generalize them. As mentioned in the previous section, in each of the known proofs [5], [6], [14], [3] the statement is proven by constructing for each graph a coloring which falsifies the Ramsey property for P_n. In the following we focus on three key tools, which are essential for obtaining such a coloring in the known proofs, namely vertex partitions, spanning trees and minimum degree arguments. For each of those tools we present occurring obstacles when considering a potential generalization to k-uniform hypergraphs.

Vertex partitions

A vertex partition of a graph G is a collection $V_1, \dots, V_i$ of pairwise disjoint vertex sets such that $V_1 \uplus \cdots \uplus V_i = V(G)$; for k-uniform hypergraphs a vertex partition is defined analogously. A key difference between considering the edges of a graph and of a k-graph with regard to such a partition is the amount of potential "placements" of an edge regarding this partition: In a graph, each edge is either completely contained in one of the partition sets or there are precisely two distinct partition sets each of which contains one vertex of the edge. In a k-graph, each edge is a k-set of vertices, so an edge possibly has multiple vertices in several different partition sets.

These various kinds of edges complicate the construction of an edge 2-coloring based on such a vertex partition. While for graphs it is possible to assign one of the colors to the edges which are entirely contained in a single partition set and the other color to all edges with one vertex in each of two different partition sets, there are insufficient colors to separately color each kind of edges in the k-uniform case.

An obvious workaround for this problem is to use the first color for the edges entirely contained in a single partition set and the second for all remaining edges. Apart from this coloring there are several other plausible constructions of a coloring based on a vertex partition. However, in the following we only focus on the mentioned example and show the resulting obstacles. In the graph case a basic observation used for showing that there is no large monochromatic path in this coloring is the following.

Observation 3.1. *Let $n \in \mathbb{N}$ and let G be a graph on at least n vertices. Let $V_1 \subseteq V(G)$ be a vertex set of size $|V_1| = \left\lfloor \frac{n}{2} \right\rfloor - 1$ and let $V_2 = V(G) \setminus V_1$. Then there is no path on n vertices in G which only consists of edges which contain a vertex in both V_1 and V_2.* $\square$

When considering the k-uniform case, a similar statement is only possibly for a substantially smaller set V_1.

Observation 3.2. *Let $n \in \mathbb{N}$, $k \geq 3$, $1 \leq \ell \leq k-1$ with $m_{n,\ell}^{(k)} \in \mathbb{N}$. Let $\mathcal{P}$ be a k-uniform ℓ-path on n vertices. Then there exists a vertex partition $V_1 \,\dot\cup\, V_2 = V(\mathcal{P})$ with $|V_1| = \left\lfloor \frac{n}{k} \right\rfloor$ such that each edge in $E(\mathcal{P})$ contains a vertex in both V_1 and V_2.*

Proof. Fix a path enumeration of the vertices $V(\mathcal{P}) = \{v_1, \ldots, v_n\}$, i.e. an ordering consecutively along the ℓ-path $\mathcal{P}$. Then let $V_1 = \left\{v_{i \cdot k} \colon i \in \left[\left\lfloor \frac{n}{k} \right\rfloor\right]\right\}$ and $V_2 = V(\mathcal{P}) \backslash V_1$. Each edge in $E(\mathcal{P})$ is a k-set of consecutive path vertices, so it contains exactly one vertex in V_1. □

As a consequence, a generalization of Observation 3.1 is only possible for a smaller set V_1 (in fact such a statement holds for $|V_1| = \left\lfloor \frac{n}{k} \right\rfloor - 1$). The size of this set is essential for the quality of the resulting bound on the size-Ramsey number, so by using this construction of a coloring based on a vertex partition a generalization of vertex partitions arguments to k-graphs is possible, but provides weaker bounds than in the graph case. Similar weaknesses also occur for the other such constructions.

Spanning trees

It is a well-known basic observation in graph theory that there exists a spanning tree in each connected graph. Such a subgraph combines several useful properties, particularly interesting for this application is the following property. A graph G is said to be *minimal connected* if G is connected but the deletion of any edge from $E(G)$ disconnects the vertices of this edge.

Observation 3.3. *Let $N \in \mathbb{N}$. Let G be a connected graph on N vertices. Then there exists a spanning, minimal connected subgraph $T \subseteq G$ which consists of $N-1$ edges.* □

It remains open in this thesis whether there is a suitable generalization of such subgraphs, i.e. spanning, minimal connected subhypergraphs that exist in each k-graph. However, there is a simple observation indicating that, if there exists such a generalization, the resulting bound on the size-Ramsey number of ℓ-paths is weaker than in the graph case. We consider the k-uniform ℓ-path on N vertices $\mathcal{P}_{N,\ell}^{(k)}$ for $k \geq 3$, $1 \leq \ell \leq k-1$ and $N \in \mathbb{N}$ with $\frac{N-\ell}{k-\ell} \in \mathbb{N}$. Then clearly every spanning subhypergraph of this k-graph consists of only at most $m_{N,\ell}^{(k)}$ edges.

While in the graph case each connected graph on N vertices has a spanning subgraph on $N-1$ edges, there are two possibilities in the generalized case: Either we say that $\mathcal{P}_{N,\ell}^{(k)}$ is not connected (which is not plausible for this application) or there exists a connected k-graph on N vertices such that every spanning subhypergraph only consists of $m_{N,\ell}^{(k)}$ many edges. If $\ell < k-1$, then $m_{N,\ell}^{(k)} < \frac{N}{2}$, which leads to significantly weaker bounds in a potential generalization.

Minimum degree arguments

In the graph case the basic idea of using the minimum degree as a tool for finding the required coloring is the following. Let G be a graph, then we partition $E(G)$ depending on whether there is a low degree vertex contained in this edge. Let

$$E_1 = \Big\{e \in E(G) \colon \exists v \in e \text{ with } deg_G(v) \leq 2\Big\} \quad \text{and} \quad E_2 = E(G) \backslash E_1.$$

Then it is possible to construct a coloring of the subgraph of G induced by E_1 such that the maximal length of a monochromatic path is 2. With some additional arguments this allows for a restriction to graphs G with minimum degree 3.

Lemma 3.4. *[3, Lemma 2.4] Let $n \in \mathbb{N}$ with $n \geq 6$. Suppose that for each graph with minimum degree 3 on at most m edges there is a coloring without a monochromatic P_{n-2}. Then each graph on at most m edges has a coloring with no monochromatic P_n.*

Such a result is a strong simplification of the problem, because in a graph G with minimum degree 3 there is an upper bound on the number of the vertices,

$$|V(G)| \leq \frac{2 \cdot e(G)}{3}.$$

A generalization of this lemma to k-graphs is not given in this thesis, instead we restrict our view to the best plausible such generalization. Using the arguments of Lemma 3.4 the theoretically best minimum degree to achieve is $\left\lceil \frac{k}{k-\ell} \right\rceil + 1$. (where $\left\lceil \frac{k}{k-\ell} \right\rceil$ is the maximum degree of a k-uniform ℓ-path). Then similarly to the graph case we obtain for a k-graph $\mathcal{G}$ with minimum degree $\left\lceil \frac{k}{k-\ell} \right\rceil + 1$,

$$|V(\mathcal{G})| \leq \frac{k \cdot e(\mathcal{G})}{\left\lceil \frac{k}{k-\ell} \right\rceil + 1}.$$

Observe that the leading factor of $e(\mathcal{G})$ is coarser than in the graph case. For $k \geq 3$ it holds that

$$\frac{k}{\left\lceil \frac{k}{k-\ell} \right\rceil + 1} \geq \frac{k}{k+1} > \frac{2}{3}.$$

The leading factor is particularly weaker if ℓ is small in terms of k. Such coarser estimates in the generalized case lead to a weaker bound on the size-Ramsey number.

As the mentioned obstacles demonstrate, a generalization of the known approaches for proving a lower bound on the size-Ramsey number of paths either does not exist or provides a weaker bound than in the graph case. Without additional refinements of these tools, it is not clear how to transfer any of the mentioned proofs so that they improve the trivial lower bound, especially not if ℓ is small in comparison to k. In the rest of this thesis we will avoid such arguments when showing lower bounds on this number.

4 Crossing approach

4.1 Introduction to the idea

This section considers an approach for proving a lower bound on the size-Ramsey number of k-uniform ℓ-paths which is referred to as the *crossing approach.* The following observation is fundamental for this approach.

Observation 4.1. *Let $n \in \mathbb{N}$, $k \geq 2$, $1 \leq \ell \leq k-1$ such that $m_{n,\ell}^{(k)} \in \mathbb{N}$. Let $\mathcal{G}$ be a k-graph with the Ramsey property $\mathcal{G} \to \mathcal{P}_{n,\ell}^{(k)}$. Then $\mathcal{G}$ contains two edge-disjoint k-uniform ℓ-paths on $m_{n,\ell}^{(k)} - 1$ edges each.*

Proof. Consider such a k-graph $\mathcal{G}$ with $\mathcal{G} \to \mathcal{P}_{n,\ell}^{(k)}$. If $\mathcal{G}$ does not contain any ℓ-path on $m_{n,\ell}^{(k)} - 1$ edges, then especially there is no $\mathcal{P}_{n,\ell}^{(k)}$ contained in $\mathcal{G}$. In this case each coloring of $\mathcal{G}$ witnesses $\mathcal{G} \nrightarrow \mathcal{P}_{n,\ell}^{(k)}$, a contradiction.

Hence, we suppose that there is such an ℓ-path on $m_{n,\ell}^{(k)} - 1$ edges. Color all its edges blue and the remaining edges of $\mathcal{G}$ red. Using the assumed Ramsey property, there is a monochromatic red $\mathcal{P}_{n,\ell}^{(k)}$ in this coloring, and therefore especially a red ℓ-path on $m_{n,\ell}^{(k)} - 1$ edges. Clearly, both ℓ-paths on $m_{n,\ell}^{(k)} - 1$ edges are edge-disjoint.

□

In the following we introduce notation that will be used throughout this section. Whenever applying Observation 4.1 to a k-graph $\mathcal{G}$ we denote the two ℓ-paths by $\mathcal{P}_1$ and $\mathcal{P}_2$. We say that the edges of $E(\mathcal{G})$ which are not contained in one of the two edge-disjoint paths $\mathcal{P}_1, \mathcal{P}_2$ are *additional* edges.

In order to prove a lower bound $\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) > f(n,k,\ell)$, where $f(n,k,\ell)$ is some positive real number, we shall find a coloring which witnesses $\mathcal{G} \nrightarrow \mathcal{P}_{n,\ell}^{(k)}$ for each k-graph $\mathcal{G}$ on at most $f(n,k,\ell)$ edges. In fact, it suffices to prove this for each k-graph $\mathcal{G}$ on precisely $\lfloor f(n,k,\ell) \rfloor$ many edges, because each k-graph on less edges can be considered as the subhypergraph of such a $\mathcal{G}$ (and $\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \in \mathbb{N}$ by definition).

When using the crossing approach on a k-graph $\mathcal{G}$, the colorings that we construct have the following property: A coloring of $\mathcal{G}$ is said to be *bad* if there exists no monochromatic red ℓ-path on n vertices in $\mathcal{G}$ and the coloring consists of precisely $m_{n,\ell}^{(k)} - 1$ blue edges. Obviously, such a coloring does not contain a monochromatic ℓ-path on n vertices of any color and so witnesses $\mathcal{G} \nrightarrow \mathcal{P}_{n,\ell}^{(k)}$.

The main obstacle in constructing a bad coloring is to assign a color to the additional edges, so those edges in neither of the two ℓ-paths $\mathcal{P}_1$ and $\mathcal{P}_2$ given by Observation 4.1. While the edges in $E(\mathcal{P}_1)$ and in $E(\mathcal{P}_2)$ each form an ℓ-path, there is no assumption on the structure of the additional edges. By that reason, we assign the same color to all additional edges and use the known structure of $E(\mathcal{P}_1)$ and $E(\mathcal{P}_2)$ to color to the remaining edges so that the resulting coloring is bad.

The starting point for this proceeding is to consider occurrences where the two ℓ-paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are "crossing" each other. A *crossing* of two ℓ-paths $\mathcal{P}_1$ and $\mathcal{P}_2$ is an ordered pair of non-empty edge sets (E_1, E_2) such that there exists a *crossing vertex set* $X \subseteq V(\mathcal{P}_1) \cap V(\mathcal{P}_2)$, $|X| = \ell$ with

$$E_1 = \Big\{e \in E(\mathcal{P}_1) \colon X \subseteq e\Big\} \quad \text{and} \quad E_2 = \Big\{e \in E(\mathcal{P}_2) \colon X \subseteq e\Big\}.$$

A property worth noting of such a crossing is that each edge $e_1 \in E_1$ and each edge $e_2 \in E_2$ form an ℓ-path on two edges. Note that two distinct crossings have distinct crossing vertex sets, and a crossing possibly has multiple crossing vertex sets. Furthermore, it might be interesting for the reader to observe that for two distinct crossings $(E_1, E_2), (E_1', E_2')$ it is possible that simultaneously $E_1' \subsetneq E_1$ and $E_2' \subsetneq E_2$ (see the example below).

Example.
Consider two 4-uniform 2-paths $\mathcal{P}_1, \mathcal{P}_2$, which have the vertices x_1, x_2 and x_3 in common as depicted below. Then $(\{e_{1,2}, e_{1,3}\}, \{e_{2,2}, e_{2,3}\})$ is a crossing using the crossing vertex set $\{x_2, x_3\}$. Additionally, $(\{e_{1,2}\}, \{e_{2,2}\})$ is a crossing with the crossing vertex set $\{x_1, x_2\}$ or alternatively $\{x_1, x_3\}$.

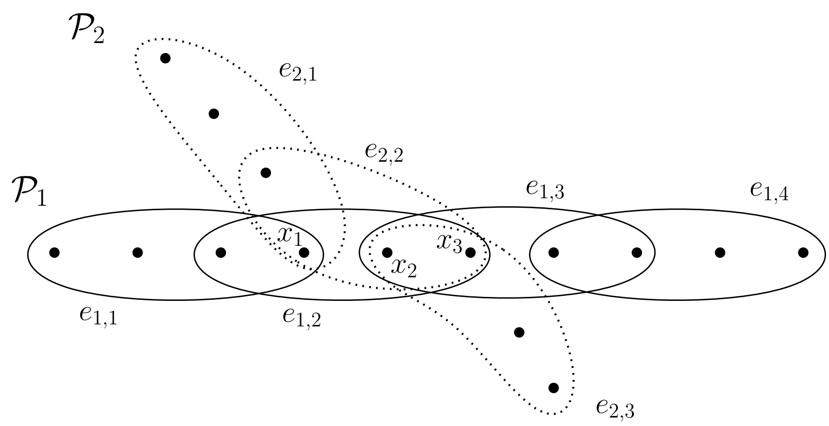

Figure 4: Example of two crossings in the case $k = 4$, $\ell = 2$

In this thesis we only consider crossings of the two ℓ-paths obtained from Observation 4.1. Working with such crossings we use some further notation. For a crossing (E_1, E_2) we say that E_1 is the set of *base edges* and E_2 is the set of *overlapping edges*. This notation can be illustrated as the ℓ-path $\mathcal{P}_1$ being overlapped by the second ℓ-path $\mathcal{P}_2$. Note that, since E_1 solely consists of path edges in the ℓ-path $\mathcal{P}_1$, there are only at most two base edges containing the entire set of ℓ crossing vertices. This implies that $1 \leq |E_1| \leq 2$ and analogously $1 \leq |E_2| \leq 2$. Moreover, observe that if $|E_1| = 2$ or $|E_2| = 2$, the crossing vertex set of a crossing is uniquely determined by the intersection of the two edges in E_1 or E_2, respectively.

Two crossings $(E_1, E_2), (E_1', E_2')$ are *base-disjoint* if their base edge sets E_1, E_1' are disjoint and *edge-disjoint* if both E_1, E_1' and E_2, E_2' are disjoint. A crossing (E_1, E_2) is *marginal* if at least one of the sets E_1, E_2 exclusively consists of an end edge of its respective ℓ-path $\mathcal{P}_1$ or $\mathcal{P}_2$. Accordingly, a crossing is *non-marginal* if both E_1 and E_2 contain an edge which is not an end edge of the respective ℓ-path.

In addition, we distinguish crossings depending on its number of base edges and overlapping edges. For $i_1, i_2 \in [2]$, we say that a crossing (E_1, E_2) is *(i_1, i_2)-structured* (or for short a *(i_1, i_2)-crossing*) if it is not marginal and both $|E_1| = i_1$ and $|E_2| = i_2$.

It is worth mentioning that the crossing approach only provides lower bounds up to $\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \geq 3m_{n,\ell}^{(k)} - 1$, since the described procedure requires that the number of additional edges is at most $m_{n,\ell}^{(k)} - 1$. Consequently, only slight improvements on the trivial bound Observation 2.9 are possible using this proof technique.

4.2 ℓ-paths for $\ell \leq \frac{k}{2}$

As a first result utilizing the crossing approach we prove a rough lower bound for the size-Ramsey number of k-uniform ℓ-paths for $1 \leq \ell \leq \frac{k}{2}$. Note that the following bound is weaker than Theorem 1.10, which will be proven in the next section.

Theorem 4.2. *Let $k \geq 2$ and $1 \leq \ell \leq \frac{k}{2}$. Let $n \in \mathbb{N}$ sufficiently large with $m_{n,\ell}^{(k)} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) > \left(2 + \frac{\ell}{4k}\right) m_{n,\ell}^{(k)} - 3.$$

Proof. Let $\mathcal{G}$ be an arbitrary k-graph $\mathcal{G}$ on precisely $\left\lfloor \left(2 + \frac{\ell}{4k}\right) m_{n,\ell}^{(k)} \right\rfloor - 3$ many edges. We shall find a bad coloring of $\mathcal{G}$, so a coloring with precisely $m_{n,\ell}^{(k)} - 1$ blue edges and the property that there exists no red $\mathcal{P}_{n,\ell}^{(k)}$ in it. Such a coloring then implies $\mathcal{G} \nrightarrow \mathcal{P}_{n,\ell}^{(k)}$.

By Observation 4.1 there are two edge-disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ on $m_{n,\ell}^{(k)} - 1$ edges each. The amount of additional edges, i.e. edges which are neither in $E(\mathcal{P}_1)$ nor in $E(\mathcal{P}_2)$, is precisely $\left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor - 1$. We proceed by firstly considering the case that there are many distinct base edges of crossings and finding a bad coloring under this assumption.

Case 1. There are at least $\frac{2(\ell+k)}{\ell} \cdot \left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor$ edges in $E(\mathcal{P}_1)$ which are the base edge in a crossing.

In this case let $B \subseteq E(\mathcal{P}_1)$ be a set of arbitrary such base edges with

$$|B| = \frac{2(\ell+k)}{\ell} \cdot \left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor .$$

Now we define a coloring and verify that it is bad. Assign the color red to each edge in B and color the following edges in blue:

(1) Each edge in $\Big(E(\mathcal{P}_1) \cup E(\mathcal{P}_2)\Big) \backslash B$ that shares at least ℓ vertices with an edge in B

(2) Each additional edge

Color the remaining unconsidered edges arbitrarily such that there are precisely $m_{n,\ell}^{(k)} - 1$ blue edges. This is possible due to the following claim.

Claim. The amount of edges considered in (1) and (2) is at most $m_{n,\ell}^{(k)} - 1$.

Proof of the claim. For estimating the edges considered in (1), fix an element $e \in B$. Since $\mathcal{P}_1$ is an ℓ-path, there are at most 2 edges in $E(\mathcal{P}_1)$ which share ℓ vertices with $e \in E(\mathcal{P}_1)$. Considering such edges in $E(\mathcal{P}_2) \backslash B$, it is an important observation that for the chosen parameter ℓ the ℓ-path $\mathcal{P}_2$ has maximum degree 2. Therefore, the k vertices which form the edge e each occur in at most two edges in $E(\mathcal{P}_2)$. Then the number of edges in $E(\mathcal{P}_2)$ which intersect e in at least ℓ vertices is at most $\frac{2k}{\ell}$.

The amount of additional edges is $\left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor - 1$, thus the total number of edges considered in (1) and (2) is at most

$$\left(2 + \frac{2k}{\ell}\right) \left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor + \left(\left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor - 1 \right) \leq \left(3 + \frac{2k}{\ell}\right) \frac{\ell}{4k} m_{n,\ell}^{(k)} - 1 \leq m_{n,\ell}^{(k)} - 1,$$

using the fact that $\frac{k}{\ell} \geq 2$ (since $\ell \leq \frac{k}{2}$). □

By this claim the above coloring is indeed well-defined. In order to show that this coloring is bad, it remains to verify that there is no monochromatic red $\mathcal{P}_{n,\ell}^{(k)}$. For this purpose, assume that there is such a red ℓ-path $\mathcal{R}$ on n vertices in $\mathcal{G}$. Clearly,

$$|B| = \frac{2(\ell+k)}{\ell} \cdot \left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor < m_{n,\ell}^{(k)} = e(\mathcal{R}),$$

thus there is an edge in $E(\mathcal{R}) \backslash B$. Then again, in the above coloring the number of red edges is precisely $\left\lfloor \left(1 + \frac{\ell}{4k}\right) m_{n,\ell}^{(k)} \right\rfloor - 2$. This implies $|E(\mathcal{R}) \backslash B| < m_{n,\ell}^{(k)}$, so there is also an edge in $E(\mathcal{R}) \cap B$.

As every edge of $\mathcal{R}$ is either in B or not in B, there especially is a set of two consecutive edge $e_1 \in B, e_2 \notin B$ in $E(\mathcal{R})$ with $|e_1 \cap e_2| = \ell$. But then e_2 is assigned the color blue in the above coloring, which is a contradiction to $e_2 \in E(\mathcal{R})$. Consequently, there is a bad coloring of $\mathcal{G}$.

Case 2. There are strictly less than $\frac{2(\ell+k)}{\ell} \cdot \left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor$ many distinct base edges of crossings.

In this case we again construct a bad coloring. Color the following edges blue:

(1) Each base edge of a crossing

(2) Each additional edge

Color the remaining edges such that there are precisely $m_{n,\ell}^{(k)} - 1$ blue edges. This is possible, since the amount of edges colored in (1) and (2) is strictly less than

$$\frac{2(\ell+k)}{\ell} \cdot \left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor + \left(\left\lfloor \frac{\ell}{4k} m_{n,\ell}^{(k)} \right\rfloor - 1 \right) \leq \left(3 + \frac{2k}{\ell} \right) \frac{\ell}{4k} m_{n,\ell}^{(k)} - 1 \leq m_{n,\ell}^{(k)} - 1.$$

As a final step, in order to find a contradiction we again assume that there is a monochromatic red ℓ-path on n vertices, denoted by $\mathcal{R}$. Each additional edge is colored blue, which implies $E(\mathcal{R}) \subseteq E(\mathcal{P}_1) \cup E(\mathcal{P}_2)$. Furthermore since $e(\mathcal{P}_1) = e(\mathcal{P}_2) = m_{n,\ell}^{(k)} - 1$, it is clear that both edge sets $E(\mathcal{R}) \cap E(\mathcal{P}_1)$ and $E(\mathcal{R}) \cap E(\mathcal{P}_2)$ are non-empty. Similarly to an argument in Case 1, we obtain two red edges $e_1 \in E(\mathcal{R}) \cap E(\mathcal{P}_1)$ and $e_2 \in E(\mathcal{R}) \cap E(\mathcal{P}_2)$ such that $|e_1 \cap e_2| = \ell$. But then there is a crossing using the crossing vertex set $e_1 \cap e_2$ which contains e_1 as a base edge and e_2 as an overlapping edge. This implies that e_1 is originally colored blue in the coloring and we reach a contradiction.

Consequently, for each k-graph $\mathcal{G}$ on precisely $\left\lfloor \left(2 + \frac{\ell}{4k})\right) m_{n,\ell}^{(k)} \right\rfloor$ many edges, there is a bad coloring. As mentioned earlier, such a coloring witnesses the Ramsey property $\mathcal{G} \nrightarrow \mathcal{P}_{n,\ell}^{(k)}$ for each $\mathcal{G}$. $\square$

It is easy to see that with more careful estimations the stated bound improves to

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) > \left(2 + \frac{\ell}{2k + 3\ell} \right) m_{n,\ell}^{(k)} - 3,$$

using the same parameters n, k, ℓ as before. Noteworthily, the steps of this proof also apply to the case $\ell > \frac{k}{2}$ with a slight modification in the above claim (For each edge $e \in B$ there are at most 3 edges in $E(\mathcal{P}_2)$ which have ℓ vertices in common with e, instead of $\frac{2k}{\ell}$ many in the version stated above.), and provide the bound

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) > \left(2 + \frac{1}{6} \right) m_{n,\ell}^{(k)} - 3.$$

Moreover, for particular choices of ℓ and k it is possible to further refine and optimize the crossing approach. For instance, in the case $\ell = \frac{k}{2}$ with k even, the following better bound holds.

Theorem 4.3. *Let $k \geq 2$ even and $\ell = \frac{k}{2}$. Let $n \in \mathbb{N}$ sufficiently large such that $m_{n,\ell}^{(k)} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) > \tfrac{9}{4} m_{n,\ell}^{(k)} - 6.$$

The proof of this result is given in the appendix.

4.3 Loose paths

In the following we prove Theorem 1.6, so a lower bound on the size-Ramsey number of loose paths, by refining the proof of Theorem 4.2 for the special case $\ell = 1$.

Proof of Theorem 1.6. The case $k = 8$ is considered below in Lemma 4.4. Now suppose that $k \geq 9$ and let $c = \max\{14, k+3\}$. Let $\mathcal{G}$ be an arbitrary k-graph with

$$e(\mathcal{G}) = \left\lfloor \left(2 + \tfrac{1}{c}\right) m_{n,1}^{(k)} \right\rfloor - 6 = 2m_{n,1}^{(k)} + \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor - 6.$$

We show that there exists a bad coloring of $\mathcal{G}$, so a coloring with precisely $m_{n,1}^{(k)} - 1$ blue edges and the property that there exists no red loose path on n vertices. As mentioned in Subsection 4.1, this suffices to prove the statement.

In this proof we repeatedly use the same approach of constructing a bad coloring. Firstly, we define a partial coloring by assigning a color to some of the edges in $E(\mathcal{G})$. Then we prove that there are at most $m_{n,1}^{(k)} - 1$ blue and at most $e(\mathcal{G}) - \left(m_{n,1}^{(k)} - 1\right) = \left\lfloor \frac{c+1}{c} m_{n,1}^{(k)} \right\rfloor - 5$ many red edges already defined. Afterwards we *complete* the partial coloring, which means that we assign colors to the remaining uncolored edges arbitrarily such that we obtain a coloring in which there are exactly $m_{n,1}^{(k)} - 1$ blue edges (and so $\left\lfloor \frac{c+1}{c} m_{n,1}^{(k)} \right\rfloor - 5$ red edges). In such a coloring we denote the edges to which a color is assigned by the partial coloring as *initial edges.* The edges to which a color was assigned by completing the coloring are referred to as *subsequent edges.*

By Observation 4.1 we suppose that there are two edge-disjoint loose paths on $m_{n,1}^{(k)} - 1$ edges each, denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$. Note that there are precisely $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor - 4$ many additional edges in $\mathcal{G}$.

We proceed by distinguishing several cases depending on the number of crossings of $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{G}$; either for some $i_1, i_2 \in [2]$ (i_1, i_2)-structured crossings appear sufficiently often or there is an upper bound on the number of all crossings. In each case we construct a bad coloring using the approach described above. Without loss of generality, we suppose that there are at least as many edges in $E(\mathcal{P}_1)$ that are the base edge of a $(1,2)$-crossing as there are edges in $E(\mathcal{P}_2)$ which are the overlapping edge of a $(2,1)$-structured crossing. (Each edge possibly occurs multiple times as such an edge but is only counted once.)

Case 1. There are at least $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(2,2)$-crossings.

In this case, consider a set M of exactly $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many such crossings. We will find a subset of M such that its elements are pairwise edge-disjoint (and not only base-disjoint). For this purpose, informally speaking, we order the $(2,2)$-crossings of M along $\mathcal{P}_2$ based on their overlapping edges and select every second crossing.

In detail, we consider the segments of $\mathcal{P}_2$, i.e. 2-sets of consecutive edges in this loose path. For these segments there is a path enumeration as introduced in Subsection 1.1. Note that for each element of M its overlapping edge set is such a segment. Using the mentioned path enumeration of the segments of $\mathcal{P}_2$ we assign to each crossing in M a natural number in $\left\{1, \ldots, \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor \right\}$ according to their order in the enumeration of its overlapping edge set. Let M' be the set of all such crossings in M to which an odd value is assigned. Then it can be seen that the crossings in M' are pairwise edge-disjoint and

$$\frac{1}{2} \cdot \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor \leq |M'| \leq \frac{1}{2} \cdot \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor + 1.$$

Now we construct a bad coloring of $\mathcal{G}$ using the set M'. We proceed by defining a partial coloring at first. For each crossing in M', color each of its base and overlapping edges in red. The color of the remaining edges is assigned later on. Clearly, there are no blue initial edges and the number of red initial edges is at most

$$\frac{4}{2c} m_{n,1}^{(k)} + 4 \leq \left\lfloor \frac{c+1}{c} m_{n,1}^{(k)} \right\rfloor - 5.$$

In a next step complete the partial coloring, i.e. color the remaining uncolored edges in $\mathcal{G}$ arbitrarily such that there are precisely $m_{n,1}^{(k)} - 1$ blue edges in $\mathcal{G}$.

We shall show that this coloring is bad. For this purpose, let $\mathcal{R}$ be a monochromatic red loose path of maximal size in $\mathcal{G}$. For each element of M', consider its two base edges and its two crossing edges. All of these four edges contain some vertex v that is a crossing vertex of this crossing. Since $\mathcal{R}$ is a loose path and therefore has maximum degree 2, at most two of the four edges corresponding to the crossing are elements of $E(\mathcal{R})$, or in other words at least two edges per crossing in M' are not in $E(\mathcal{R})$. Remember that these crossings in M' are pairwise edge-disjoint. Consequently, there are at least $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many red edges not used in $\mathcal{R}$. Especially, the number of edges in $E(\mathcal{R})$ is at most

$$e(\mathcal{R}) \leq \left(\left\lfloor \frac{c+1}{c} m_{n,1}^{(k)} \right\rfloor - 5 \right) - \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor = m_{n,1}^{(k)} - 5,$$

which implies that there is no red $\mathcal{P}_{n,\ell}^{(k)}$, so the constructed coloring is bad.

Case 2. There are at least $2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(1,2)$-crossings.

We proceed similarly to Case 1 and as a consequence the following is only sketched. Firstly, let M be a set consisting of precisely $2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many such crossings. As before, we obtain a set M' of pairwise edge-disjoint $(1,2)$-crossings of size

$$\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor \leq |M'| \leq \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor + 1.$$

In a next step we construct a bad coloring. Assign the color red to each edge corresponding to a crossing in M' (so the single base edge and both overlapping edges). The number of such edges is most

$$3\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor + 3 \leq \left\lfloor \frac{c+1}{c} m_{n,1}^{(k)} \right\rfloor - 5.$$

Then complete the partial coloring as before.

Let $\mathcal{R}$ be a maximal monochromatic red loose path in $\mathcal{G}$. Once more, we observe that for each of the elements in M' only two of the three edges corresponding to the respective crossing are used in $\mathcal{R}$. We conclude that $e(\mathcal{R}) \leq m_{n,1}^{(k)} - 5$ edges and consequently there is a bad coloring in this case.

Before considering the remaining cases, we make some observations on the number of edges involved in crossings that are $(1,2)$-, $(2,1)$- or $(2,2)$-structured under the condition that neither Case 1 nor Case 2 holds, so that there neither exists a set of $2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(1,2)$-crossings nor a set of $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ pairwise base-disjoint $(2,2)$-crossings.

Regarding $(1,2)$-crossings we know that each such crossings only has a single base edge. Hence, if Case 2 does not hold, there are at most $2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many edges in $E(\mathcal{P}_1)$ which are a base edge of a $(1,2)$-crossing. Remember that we supposed that the number of such edges is larger or equal than the number of edges in $E(\mathcal{P}_2)$ which are the overlapping edge in a $(2,1)$-crossing. Consequently, the latter number is also bounded by $2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$.

Furthermore, we estimate the number of base edges in $(2,2)$-crossings.

Claim. There are at most $\left\lfloor \frac{3}{c} m_{n,1}^{(k)} \right\rfloor$ many edges in $E(\mathcal{P}_1)$ which are a base edge in a $(2,2)$-crossing.

Proof of the claim. Let B be the set of all base edges of $(2,2)$-crossings. In order to estimate the size of B, consider a maximal sized set M consisting of pairwise base-disjoint $(2,2)$-crossings and let $B_M \subseteq E(\mathcal{P}_1)$ be the set of all their base edges. It clearly holds that $|B_M| = 2|M| \leq 2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$, so it remains to examine the set $B \backslash B_M$.

Consider an arbitrary $(2,2)$-crossing Φ which has a base edge $e_1 \in B\backslash B_M$. By maximality of M, we know that the second base edge of Φ, say e_2, is contained in B_M. Especially, there is a unique crossing $\Psi_\Phi \in M$ which also has e_2 as a base edge. Let e_3 be the other base edge of Ψ_Φ.

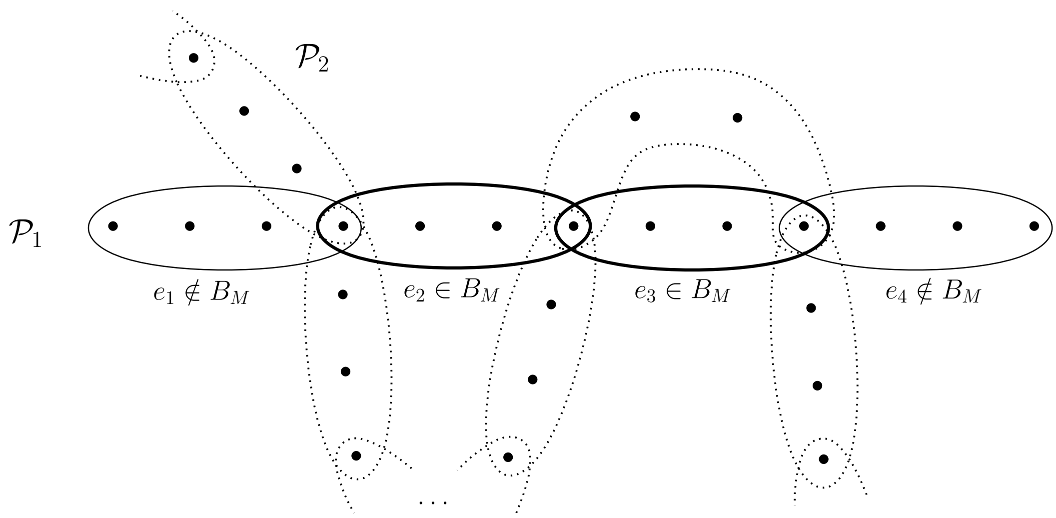

Figure 5: The edges e_1, e_2, e_3 and e_4 which form the base edge sets of Φ, Ψ_Φ and Φ'

We want to show that except for Φ there is no second $(2,2)$-crossing which has one base edge in common with Ψ_Φ and one base edge outside of B_M. So assume for a contradiction that there exists a $(2,2)$-crossing Φ' with $\Phi' \neq \Phi$ such that Φ' has a base edge $e_4 \in B\backslash B_M$ and the unique crossing in M which shares a base edge with Φ' is Ψ_Φ. Then $e_4 \neq e_1$ and using the fact that the base edges of a crossing are consecutive in $\mathcal{P}_1$ we obtain that the edge e_2 is not a base edge of Φ'. This implies that the base edges of Φ' are e_3 and e_4. Therefore Φ and Φ' are base-disjoint, as illustrated above.

For both, Φ and Φ', one of the two base edges is in $B\backslash B_M$ while the second is a base edge of Ψ_Φ. This implies that Φ and Φ' are both base-disjoint from every crossing in $M\backslash\{\Psi_\Phi\}$. Under this condition, replacing Ψ_Φ in M by Φ and Φ' increases the size of M while maintaining the property of being pairwise base-disjoint. This contradicts the maximality of M.

As a consequence, we find an injective correspondence between the elements of $B\backslash B_M$ and the crossings in M. Each edge in $B\backslash B_M$ is the base edge of some $(2,2)$-crossing Φ with a base edge in $B\backslash B_M$. Since Φ also has a base edge in B_M, there is no second edge in $B\backslash B_M$ corresponding to the same Φ. Moreover, we have shown that the crossing $\Psi_\Phi \in M$ solely correspond to this Φ and to no other such crossing. Thus, we obtain $|B\backslash B_M| \leq |M|$ and consequently,

$$|B| = |B_M| + |B\backslash B_M| \leq 3|M| \leq \left\lfloor \frac{3}{c} m_{n,1}^{(k)} \right\rfloor.$$

□

We now use this claim in order to prove the remaining cases. Note that the procedure in the following Cases 3 and 4 is roughly similar to the Cases 1 and 2 in Theorem 4.2.

Case 3. There are at least $\left\lfloor \frac{6}{c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(1,1)$-crossings.

Let B be the set consisting of all edges in $E(\mathcal{P}_1)$ which are the base edge of a $(1,1)$-structured crossing. Since each $(1,1)$-crossing has only a single base edge, it can be seen that $|B| \geq \left\lfloor \frac{6}{c} m_{n,1}^{(k)} \right\rfloor$. Applying the previous assumptions and the above claim we see that there are at most $\left\lfloor \frac{5}{c} m_{n,1}^{(k)} \right\rfloor$ edges in $E(\mathcal{P}_1)$ which are a base edge in a $(1,2)$-structured or a $(2,2)$-structured crossing. Especially, there is a subset $B' \subseteq B$ of size $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ such that each element of B' is the base edge of a $(1,1)$-crossing but neither of a $(1,2)$- nor a $(2,2)$-crossing.

Once more, we construct a bad coloring by primarily defining a partial coloring. Color each element of B' in red and color the following edges in blue:

(1) Each edge in $\Big(E(\mathcal{P}_1) \cup E(\mathcal{P}_2)\Big)\backslash B'$ which shares a vertex with an edge in B'

(2) Each additional edge

In order to estimate the number of edges colored in (1), fix an element $e \in B'$. Note that e is not an end edge of $\mathcal{P}_1$ since it is the single base edge of a non-marginal crossings.

There are at most two edges in $E(\mathcal{P}_1)\backslash\{e\}$ which have a vertex in common with e, the two edges consecutive to e in $\mathcal{P}_1$. Considering $E(\mathcal{P}_2)$, there are possibly various edges that share a vertex with e. Note that $\mathcal{P}_2$ is a k-graph with maximum degree 2, so each vertex $v \in e$ is only in at most two edges of $\mathcal{P}_2$. In addition to that, if there is a vertex $v \in e$ such that v is contained in two edges in $E(\mathcal{P}_2)$, then $\{v\}$ is the crossing vertex set of a crossing which has e as a base edge and the mentioned two edges in $E(\mathcal{P}_2)$ as overlapping edges. Especially, this crossing is non-marginal and by this $(1,2)$- or $(2,2)$-structured. This contradicts $e \in B'$, so we deduce that there each vertex $v \in e$ is contained in at most one edge of $\mathcal{P}_2$. Thus there are at most k edges in $E(\mathcal{P}_2)$ which share a vertex with e.

We obtain that the number of blue initial edges is bounded by

$$\Big(2+k\Big) \cdot \frac{1}{c} m_{n,1}^{(k)} + \left(\frac{1}{c} m_{n,1}^{(k)} - 4\right) < m_{n,1}^{(k)}, \quad \text{since } c \geq k+3.$$

Additionally, the number of red edges in the partial coloring is exactly $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$. Then complete the partial coloring so that we obtain a coloring with precisely $m_{n,\ell}^{(k)} - 1$ many blue edges.

Now assume for a contradiction that there is a red loose path $\mathcal{R}$ on n vertices in $\mathcal{G}$. Observe that the red initial edges only have vertices in common with other initial edges (possibly red or blue ones). Especially, there is no red subsequent edge intersecting with a red initial edge. Thus, $\mathcal{R}$ uses either exclusively initial edges or exclusively subsequent edges. As there are exactly $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ red initial edges, there exist at most $m_{n,\ell}^{(k)} - 5$ red subsequent edges. So in either of the two cases, it follows that $e(\mathcal{R}) < m_{n,\ell}^{(k)}$, which is a contradiction. This implies that the coloring is bad.

Case 4. Neither of the Cases 1, 2, 3 holds, so there neither exist a set of $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(2,2)$-crossing, nor a set of $2\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ base edges of $(1,2)$-crossings, nor $\left\lfloor \frac{6}{c} m_{n,1}^{(k)} \right\rfloor$ distinct base edges of $(1,1)$-crossings.

Similarly before, we define a partial coloring. There are no edges which initially are colored red. The following edges are colored blue:

(1) Each base edge of a $(1,1)$-crossing

(2) Each base edge of a $(1,2)$-crossing

(3) Each crossing edge of a $(2,1)$-crossing

(4) Each base edge of a $(2,2)$-crossing

(5) Each additional edge

(6) Each of the end edges of the loose paths $\mathcal{P}_1$ and $\mathcal{P}_2$

Using the above claim and the condition of the case, we obtain that the number of blue initial edges is strictly less than

$$\left(6+2+2+3\right) \cdot \frac{1}{c} m_{n,1}^{(k)} + \left(\frac{1}{c} m_{n,1}^{(k)} - 4\right) + 4 \leq m_{n,\ell}^{(k)}, \quad \text{since } c \geq 14.$$

Equivalently, there are at most $m_{n,1}^{(k)} - 1$ blue initial edges (since $m_{n,1}^{(k)} \in \mathbb{N}$).

Complete the coloring by assigning arbitrary colors to the remaining uncolored edges such that there are precisely $m_{n,1}^{(k)} - 1$ many blue edges in $\mathcal{G}$. We shall prove that there is no monochromatic red $\mathcal{P}_{n,1}^{(k)}$ in $\mathcal{G}$. For this purpose, let $R \subseteq E(\mathcal{G})$ be the set of all red edges. We partition R according to the two loose paths $\mathcal{P}_1, \mathcal{P}_2$, i.e. let $R_1 = R \cap E(\mathcal{P}_1)$ and $R_2 = R \cap E(\mathcal{P}_2)$. Note that $R = R_1 \cup R_2$ as every additional edge is colored blue.

For each crossing in $\mathcal{G}$ either all base edges or all overlapping edges are colored blue (For marginal crossings this is witnessed by the end edges in (6).), so in the red subhypergraph of $\mathcal{G}$ there is no crossing and consequently there are no red edges $e_1 \in E(\mathcal{P}_1)$, $e_2 \in E(\mathcal{P}_2)$ which have a vertex in common. Especially, each edge in R_1 is disjoint from each edge in R_2, so a monochromatic red loose path consists of edges either completely in R_1 or completely in R_2. It is clear that $|R_1| \leq e(\mathcal{P}_1) = m_{n,1}^{(k)} - 1$ and $|R_2| \leq e(\mathcal{P}_2) = m_{n,1}^{(k)} - 1$, so there is no monochromatic red loose path on $m_{n,1}^{(k)}$ edges in this coloring.

Hence, the constructed coloring is bad, especially there exists a bad coloring for each k-graph $\mathcal{G}$ on $\left\lfloor \left(2 + \frac{1}{c}\right) m_{n,1}^{(k)} - 6 \right\rfloor$ edges, which concludes the proof. □

It might be interesting to the reader that the bottleneck of this proof, i.e. the bound which is decisive for the quality of the result, is the estimate on the blue initial edges in Case 3, if $k \geq 11$, and the estimate in Case 4, if $k \leq 11$. (For $k = 11$ both bounds are bottlenecks.)

Noteworthily, the mentioned estimate in Case 3 is is the only part of the proof where it is required that the number of additional edges is – speaking asymptotically in k – only at most $O_k\left(\frac{1}{k}\right) m_{n,1}^{(k)}$ while the other bounds also hold for $\alpha \cdot m_{n,1}^{(k)}$ many additional edges for some leading factor $\alpha \in \mathbb{R}, \alpha > 0$ independent from k. However, it remains open in this thesis how to overcome this bottleneck.

The above proof can also be applied for the case $2 \leq k \leq 8$ (using $c = 14$ in the proof), which for $n \in \mathbb{N}$ sufficiently large with $m_{n,1}^{(k)} \in \mathbb{N}$ yields the bound

$$\hat{R}^{(k)}(\mathcal{P}_{n,1}^{(k)}) \geq \left(2 + \frac{1}{14}\right) m_{n,1}^{(k)} - 6,$$

a weaker result than Theorem 1.5. However, there are further refinements of the previous proof for small values of k. Note that the case $k = 8$ of the following result completes the proof of Theorem 1.6.

Lemma 4.4. *Let $2 \leq k \leq 8$ and $n \in \mathbb{N}$ sufficiently large with $m_{n,1}^{(k)} \in \mathbb{N}$. Then*

$$\hat{R}^{(k)}(\mathcal{P}_{n,1}^{(k)}) > \left(2 + \frac{1}{c(k)}\right) m_{n,1}^{(k)} - 6$$

where $c(k)$ is given by the table below.

k	2	3	4	5	6	7	8
$c(k)$	4	$\frac{17}{2}$	9	11	11	12	13

Proof. In the case $k = 2$ every non-marginal crossing consists of precisely two base edges and two overlapping edges, so there are no crossings which are $(1,1)$-, $(1,2)$- or $(2,1)$-structured. Performing the same proof as above (using $c = 4$), but omitting Case 2 and Case 3 leads to the required bound.

For $3 \leq k \leq 8$, the main part of the proof is executed analogously to before, there is only a modification of Case 3. Note that by changing the bound in the condition of Case 3 the arguments in the remaining proof stay the same despite that there are different numbers in Case 4. In the following we therefore only present the new version of Case 3 as well as the resulting bound in Case 4. Given $\mathcal{G}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ as before and $c = c(k)$, we introduce a Case 3′ for $5 \leq k \leq 8$ and Case 3″ for $k \in \{3, 4\}$ as a replacement for Case 3 of the above proof. For both of these new cases we show that there exists a bad coloring in $\mathcal{G}$.

Firstly, let $5 \leq k \leq 8$ and consider the following condition.

Case 3′. There are at least $\left\lfloor \frac{3}{c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(1,1)$-crossings.

Some parts of the following are only sketched as they are similar to the proceeding in Case 3 of the previous proof. Let B be the set of edges in $E(\mathcal{P}_1)$ which are the base edge of a $(1,1)$-crossing. Then we find a subset $B' \subseteq B$ of size precisely $\left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$ such that each element of B' is the base edge of a $(1,1)$-crossing but not of a $(1,2)$-crossing. (Unlike in the original Case 3 it is allowed for an edge in B' to be a base edge of a $(2,2)$-structured crossing.)

The partial coloring we construct is defined as before, so each element of B' is colored red and the following edges are colored blue:

(1) Each edge in $\Big(E(\mathcal{P}_1) \cup E(\mathcal{P}_2)\Big) \backslash B'$ which shares a vertex with an edge in B'

(2) Each additional edge

In order to estimate the number of edges considered in (1), fix some $e \in B'$. As before, there are at most two edges in $E(\mathcal{P}_1) \backslash \{e\}$ which have a vertex in common with e. Regarding $E(\mathcal{P}_2)$, we distinguish the vertices in e depending on their degree in $\mathcal{P}_1$.

For each vertex in e of degree 1 in $\mathcal{P}_1$, there is at most one edge in $E(\mathcal{P}_2)$ containing that vertex since e is not the base edge of a $(1,2)$-crossing.

For each vertex of e with degree 2 in $\mathcal{P}_1$, there are at most two edges in $E(\mathcal{P}_2)$ in which that vertex is contained (unlike in the original Case 3).

It can be seen that e is not an end edge of $\mathcal{P}_1$ as it is the single base edge of a non-marginal crossing. Since $\mathcal{P}_1$ is a loose path, the edge e consists of two vertices of degree 2 and $k - 2$ vertices of degree 1 in $\mathcal{P}_1$. Consequently, the following upper bound for the number of blue initial edges holds,

$$\Big(2 + (k-2) + 4\Big) \cdot \frac{1}{c} m_{n,1}^{(k)} + \left(\frac{1}{c} m_{n,1}^{(k)} - 4\right) < m_{n,1}^{(k)}, \qquad \text{since } c \geq k+5.$$

As before, the completion of this partial coloring is bad.

Using Case 3′ instead of Case 3 in the proof above leads to the required bound. Note that in Case 4 we obtain that the number of blue initial edges is strictly less than

$$\Big(3 + 2 + 2 + 3\Big) \cdot \frac{1}{c} m_{n,1}^{(k)} + \left(\frac{1}{c} m_{n,1}^{(k)} - 4\right) + 4 \leq m_{n,\ell}^{(k)}, \qquad \text{since } c \geq 11.$$

It remains to consider the case that $k \in \{3, 4\}$. We show that there exists a bad coloring under the following condition.

Case 3″. There are at least $\left\lfloor \frac{k-2}{2c} m_{n,1}^{(k)} \right\rfloor$ many pairwise base-disjoint $(1,1)$-crossings.

Let M be an arbitrary set consisting of precisely $\left\lfloor \frac{k-2}{2c} m_{n,1}^{(k)} \right\rfloor$ pairwise base-disjoint $(1,1)$-crossings. Observe that each edge in $E(\mathcal{P}_2)$ appears at most $k-2$ times as an overlapping edge of a $(1,1)$-crossing in M. By this argument, we find a subset $M' \subseteq M$ consisting of precisely $\left\lfloor \frac{1}{2c} m_{n,1}^{(k)} \right\rfloor$ many pairwise edge-disjoint $(1,1)$-crossings. (Notably for $k = 3$ we have $M' = M$.)

As before, we define a partial coloring. Color all base and overlapping edges of crossings in M' red. Furthermore, assign the color blue to the following edges:

(1) Each further edge in $E(\mathcal{P}_1) \cup E(\mathcal{P}_2)$ which shares at least one vertex with a base edge or overlapping edge of a crossing in M'

(2) Each additional edge

Fix a $(1,1)$-crossing $(\{e_1\}, \{e_2\}) \in M'$ and let $\{v\} \subseteq V(\mathcal{G})$ be a crossing vertex set of this crossing. Consider the edges in $\Big(E(\mathcal{P}_1) \cup E(\mathcal{P}_2)\Big) \backslash \{e_1, e_2\}$ which have at least one vertex in common with e_1. As seen before, there are at most two edges in $E(\mathcal{P}_1)$ which intersect with e_1. The vertex v is not contained in any edge in $E(\mathcal{P}_2) \backslash \{e_1, e_2\}$ witnessed by the crossing $(\{e_1\}, \{e_2\})$. Every other vertex of e_1 is contained in at most two edges of $E(\mathcal{P}_2)$.

An analogous observation holds for the overlapping edge e_2. Summarizing these observations we obtain that the number of blue initial edges is at most

$$\Big(2 + 2(k-1)\Big) \cdot \frac{2}{2c} m_{n,1}^{(k)} + \left(\frac{1}{c} m_{n,1}^{(k)} - 4\right) = \frac{2k+1}{c} m_{n,1}^{(k)} - 4 < m_{n,1}^{(k)}, \qquad \text{since } c \geq 2k+1,$$

whereas the number of red-colored initial edges is $2 \left\lfloor \frac{1}{2c} m_{n,1}^{(k)} \right\rfloor \leq \left\lfloor \frac{1}{c} m_{n,1}^{(k)} \right\rfloor$.

Complete the partial coloring such that there are exactly $m_{n,1}^{(k)} - 1$ blue edges. Then the red initial edges in $\mathcal{G}$ only have vertices in common with other red or blue initial edges and the amount of these red initial edges is at least $2\left\lfloor \frac{1}{2c} m_{n,1}^{(k)} \right\rfloor \geq \frac{1}{c} m_{n,1}^{(k)} - 2$. As before, this implies that the coloring is bad.

Proceeding analogously to the proof of Theorem 1.6, only using Case 3′′ instead of Case 3, provides the required bound. Note that in the process in Case 4 the number of blue initial edges is strictly less than

$$\left(\frac{k-2}{2} + 2 + 2 + 3\right) \cdot \frac{1}{c} m_{n,1}^{(k)} + \left(\frac{1}{c} m_{n,1}^{(k)} - 4\right) + 4 \leq m_{n,\ell}^{(k)}, \qquad \text{since } c \geq 7 + \tfrac{k}{2}.$$

□

5 Neighborhood approach

5.1 Introduction to the idea

The second proof concept that is utilized in this thesis in order to prove a lower bound on the size-Ramsey number of k-uniform ℓ-paths is referred to as the *neighborhood approach*. In this subsection the basic idea and notation of the approach are introduced, while in the following subsections proofs using this proof technique are presented. A central role in this approach plays the following definition. Let $\mathcal{G}$ be a k-graph and $Z \subseteq E(\mathcal{G})$ be an edge set. For $q \in \mathbb{R}$ with $0 < q \leq k$, the $\geq q$*-neighborhood* of Z is the edge set

$$N_{\geq q}(Z) = \Big\{ e \in E(\mathcal{G}) \colon \exists e' \in Z \text{ with } |e \cap e'| \geq q \Big\}.$$

Given $\mathcal{G}$ and Z as above and for $q \in \mathbb{R}$, $0 \leq q < k$, we also introduce the $>q$*-neighborhood* of Z, which is

$$N_{>q}(Z) = \Big\{ e \in E(\mathcal{G}) \colon \exists e' \in Z \text{ with } |e \cap e'| > q \Big\}.$$

Note that in both definitions we allow that $e = e'$, so especially we have $Z \subseteq N_{\geq q}(Z)$ and $Z \subseteq N_{>q}(Z)$ for all suitable choices of q. Furthermore, it is easy to see that $N_{>q}(Z_1) \cup N_{>q}(Z_2) = N_{>q}(Z_1 \cup Z_2)$ for all $Z_1, Z_2 \subseteq E(\mathcal{G})$.

The idea of the neighborhood approach is the following. In order to show a lower bound, we consider an arbitrary k-uniform hypergraph $\mathcal{G}$ with the Ramsey property $\mathcal{G} \to \mathcal{P}^{(k)}_{n,\ell}$ (without an assumption on the number of edges in $\mathcal{G}$). We iteratively find pairwise edge-disjoint ℓ-paths in this k-graph, which provides a lower bound on the number of edges in $\mathcal{G}$. In each iteration step we choose large subpaths in the previously constructed ℓ-paths and consider the $>q$-neighborhood (or situationally the $\geq q$-neighborhood) of the edges in the selected subpaths. With a careful choice of the parameter q (depending on the iteration step) this provides a coloring of $\mathcal{G}$, which then yields a new ℓ-path edge-disjoint from those neighborhoods.

For working with such neighborhoods, the following proposition will be useful.

Proposition 5.1. *Let $k \geq 2$, $1 \leq \ell \leq k-1$ and $n \in \mathbb{N}$ with $m^{(k)}_{n,\ell} \in \mathbb{N}$. Let $\mathcal{P}$ be a k-uniform ℓ-path on n vertices. Furthermore, let $\alpha \in \mathbb{R}$, $1 \leq \alpha \leq k$ be a constant and $W \subseteq V(\mathcal{P})$ be a vertex set such that for each edge $e \in E(\mathcal{P})$ we have $|e \cap W| \geq \alpha$. Then*

$$|W| \geq \frac{\alpha(n-\ell)}{k}, \qquad \textit{if } \tfrac{k}{k-\ell} \in \mathbb{N} \textit{ and}$$

$$|W| \geq \frac{\alpha(n-\ell)}{2k-\ell-1}, \qquad \textit{if } \tfrac{k}{k-\ell} \notin \mathbb{N}.$$

Proof. We estimate the size of W by double-counting ordered pairs (v, e) consisting of a vertex $v \in W$ and an edge $e \in E(\mathcal{P})$ with $v \in e$. Let $\rho_{(v,e)}$ be the number of such pairs.

Considering the edges of $\mathcal{P}$, the following inequation is immediate.

$$\rho_{(v,e)} \geq \alpha \cdot m_{n,\ell}^{(k)} = \frac{\alpha(n-\ell)}{k-\ell}.$$

Now consider the vertices in $W \subseteq V(\mathcal{P})$. It is a simple observation that the maximum degree of a vertex in W regarding the ℓ-path $\mathcal{P}$ is precisely $\left\lceil \frac{k}{k-\ell} \right\rceil$. By this,

$$\rho_{(v,e)} \leq |W| \cdot \left\lceil \frac{k}{k-\ell} \right\rceil.$$

Combining both inequations, we obtain

$$|W| \geq \frac{1}{\left\lceil \frac{k}{k-\ell} \right\rceil} \cdot \frac{\alpha(n-\ell)}{k-\ell}.$$

If $\frac{k}{k-\ell} \in \mathbb{N}$, this provides the bound

$$|W| \geq \frac{\alpha(n-\ell)}{k}.$$

Otherwise, the following coarser bound holds,

$$|W| \geq \frac{1}{\frac{k+(k-\ell-1)}{k-\ell}} \cdot \frac{\alpha(n-\ell)}{k-\ell} \geq \frac{\alpha(n-\ell)}{2k-\ell-1}.$$

□

For arguments using the $>q$-neighborhood instead of the $\geq q$-neighborhood, we consider two implications of Proposition 5.1.

Corollary 5.2. *Let $k \geq 2$, $n \in \mathbb{N}$ and let $\mathcal{P}$ be a k-uniform tight path on n vertices. Let $W \subseteq V(\mathcal{P})$ such that for each edge $e \in E(\mathcal{P})$ we have $|e \cap W| > \frac{k+1}{2}$. Then for sufficiently large n*

$$|W| > \frac{n}{2}.$$

Proof. For each edge $e \in E(\mathcal{P})$ it holds that $|e \cap W| \geq \frac{k+2}{2}$. By Proposition 5.1 we obtain the bound

$$|W| \geq \frac{k+2}{2} \cdot \frac{n-\ell}{k} > \frac{n}{2}$$

for sufficiently large n. □

Corollary 5.3. *Let $k \geq 2$, $\frac{2}{3}k < \ell \leq k-1$, $n \in \mathbb{N}$ with $m^{(k)}_{n,\ell} \in \mathbb{N}$. Let $\mathcal{P}$ be a k-uniform ℓ-path on n vertices. Let $W \subseteq V(\mathcal{P})$ such that for each edge $e \in E(\mathcal{P})$ we have $|e \cap W| > \frac{k+1}{2}$. Then for sufficiently large n*

$$|W| > \frac{n}{3}.$$

Proof. As before, we know for each edge $e \in E(\mathcal{P})$ that $|e \cap W| \geq \frac{k+2}{2}$. Now Proposition 5.1 provides the bound

$$|W| \geq \frac{k+2}{2} \cdot \frac{n-\ell}{2k-\ell-1} \geq \frac{k+2}{2} \cdot \frac{n-\ell}{\frac{4}{3}k-1} \geq \frac{3(k+2)}{8\left(k-\frac{3}{4}\right)}(n-\ell) > \frac{n}{3}$$

for sufficiently large n. □

5.2 Tight paths

The neighborhood approach is firstly applied for proving a lower bound on the size-Ramsey number of tight paths, Theorem 1.8.

Proof of Theorem 1.8. Let $\mathcal{G}$ be a k-uniform hypergraph with $\mathcal{G} \to \mathcal{P}^{(k)}_{n,k-1}$, i.e. such that each coloring contains a monochromatic k-uniform tight path on n vertices. We show that there are there are at least $\lceil \log_2(k+1) \rceil \cdot m^{(k)}_{n,k-1} - k^2$ many edges in $\mathcal{G}$.

As mentioned in the previous subsection, we use an iterative argument to prove this statement. Let $\lambda = \lceil \log_2(k+1) \rceil - 1$ be a natural number, this number indicates how many iteration steps are executed. At first, we define a function $q\colon \{0, \dots, \lambda\} \to \mathbb{R}$,

$$q(i) = \left(1 - \frac{1}{2^i}\right)(k+1),$$

which is the parameter of the considered $>q$-neighborhoods in each iteration step.

Clearly, q is an increasing function and $q(i) \geq 0$ for $i \in \{0, \dots, \lambda\}$. Remember that for $q(i) \geq k$ the $>q(i)$-neighborhood is not defined. By that reason, note that $i \leq \lambda$ (or equivalently $i < \log_2(k+1)$) provides $q(i) < k$, which then implies that the $>q(i)$-neighborhood is indeed defined for all $i \in \{0, \dots, \lambda\}$. Apart from that, it might be interesting to the reader that possibly $q(i) \geq k-1$, which yields $N_{>q(i)}(Z) = Z$ for each edge set $Z \subseteq E(\mathcal{G})$. This special case does not affect the proof.

In the following we start to construct the mentioned tight paths. Initially, it is clear to see that there is some tight path on n vertices in $\mathcal{G}$ because otherwise each coloring witnesses $\mathcal{G} \nrightarrow \mathcal{P}^{(k)}_{n,k-1}$. We fix one such tight path and denote it by $\mathcal{P}_0$.

From now on we proceed iteratively, so let $i = 1, \dots, \lambda$ and suppose that the iteration has been performed for all smaller values of i. In each step of the iteration we construct the following:

- Edge sets $Z_i^1, Z_i^2 \subseteq E(\mathcal{P}_{i-1})$ such that $\cup Z_i^1 \cap \cup Z_i^2 = \varnothing$ and each of the sets induces a tight path in $\mathcal{G}$ on precisely $\left\lceil \frac{n}{2} \right\rceil$ vertices.
- A tight path $\mathcal{P}_i$ on n vertices with $E(\mathcal{P}_i) \cap N_{>q(i)}(Z_b^a) = \varnothing$ for all $a \in [2], b \in [i]$.

As a first step, we construct Z_i^1 and Z_i^2 by dividing the tight path $\mathcal{P}_{i-1}$ into two parts of equal length and considering the edge sets of the two created shorter tight paths. For this purpose, consider a path enumeration of the vertices $V(\mathcal{P}_{i-1}) = \{v_1, \dots, v_n\}$ as introduced in Subsection 1.1. Let

$$V_i^1 = \left\{ v_1, \dots, v_{\left\lfloor \frac{n}{2} \right\rfloor} \right\} \quad \text{and} \quad V_i^2 = \left\{ v_{\left\lceil \frac{n}{2} \right\rceil + 1}, \dots, v_n \right\}.$$

It can be seen that $|V_i^1| = \left\lfloor \frac{n}{2} \right\rfloor = |V_i^2|$. Now let $Z_i^1 = E(\mathcal{P}_{i-1}[V_i^1])$ and $Z_i^2 = E(\mathcal{P}_{i-1}[V_i^2])$. Clearly, these two sets induce vertex-disjoint tight paths on $\left\lfloor \frac{n}{2} \right\rfloor$ vertices in $\mathcal{G}$. Thus, the number of edges in Z_i^1 and in Z_i^2 is

$$|Z_i^1| = |Z_i^2| = m^{(k)}_{\left\lfloor \frac{n}{2} \right\rfloor, k-1} = \left\lfloor \frac{n}{2} \right\rfloor - k + 1 \geq \frac{n - 2k + 1}{2}.$$

In a next step we show a key property of the edge sets Z_b^a for $a \in [2]$, $b \in [i]$.

Claim 1. Let $a_1, a_2 \in [2]$, $b_1, b_2 \in [i]$ such that at least one of $a_1 \neq a_2$ or $b_1 \neq b_2$ holds (so excluding the case $a_1 = a_2 \wedge b_1 = b_2$). Then for each two edges $e_1 \in N_{>q(i)}(Z_{b_1}^{a_1})$ and $e_2 \in N_{>q(i)}(Z_{b_2}^{a_2})$ we have

$$|e_1 \cap e_2| < k - 1.$$

Proof of Claim 1. Assume that there are edges $e_1 \in N_{>q(i)}(Z_{b_1}^{a_1})$, $e_2 \in N_{>q(i)}(Z_{b_2}^{a_2})$ with $|e_1 \cap e_2| \geq k - 1$. By definition, there is an edge $z_1 \in Z_{b_1}^{a_1}$ such that $|e_1 \cap z_1| > q(i)$ and an edge $z_2 \in Z_{b_2}^{a_2}$ with $|e_2 \cap z_2| > q(i)$.

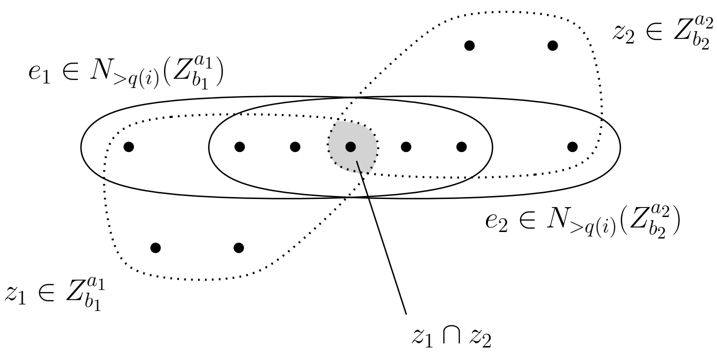

Figure 6: Possible constellation of the edges in iteration step $i = 1$ where $k = 6$

We estimate the size of $z_1 \cap z_2$ in order to find a contradiction to our assumption. Since $|e_1 \cap e_2| \geq k-1$, we have $|e_1 \backslash e_2| \leq 1$ and so $|e_2 \cap z_1| > q(i)-1$. Then the following estimate holds

$$\begin{aligned}|z_1 \cap z_2| \geq |e_2 \cap z_1 \cap z_2| \geq |e_2| - |e_2 \backslash z_1| - |e_2 \backslash z_2| &= -|e_2| + |e_2 \cap z_1| + |e_2 \cap z_2| \\ > -k + q(i) - 1 + q(i) &= \left(1 - \frac{1}{2^{i-1}}\right)(k+1) = q(i-1).\end{aligned}$$

Now consider the relation of b_1 and b_2. If $b_1 = b_2$, the edge sets $Z_{b_1}^{a_1}$ and $Z_{b_2}^{a_2}$ induce two vertex-disjoint k-graphs. Then especially $|z_1 \cap z_2| = 0 = q(0) \leq q(i-1)$, which is a contradiction.

Thus, we suppose that $b_1 \neq b_2$ and without loss of generality $b_1 > b_2$ (and by this $1 < b_1 \leq i$). By construction, $Z_{b_1}^{a_1} \subseteq E(\mathcal{P}_{b_1-1})$ and in the iteration step $b_1 - 1$ the tight path $\mathcal{P}_{b_1-1}$ was chosen edge-disjoint from $\bigcup_{a\in[2],b<b_1} N_{>q(b_1-1)}(Z_b^a)$. This yields that $z_1 \notin N_{>q(b_1-1)}(Z_{b_2}^{a_2})$ and so

$$|z_1 \cap z_2| \leq q(b_1-1) \leq q(i-1),$$

since q is an increasing function. This is a contradiction to $|z_1 \cap z_2| > q(i-1)$. □

Now we find the next tight path $\mathcal{P}_i$ in $\mathcal{G}$ by considering the following coloring of $\mathcal{G}$. For all $a \in [2]$ and $b \in [i]$, assign the color red to each edge in $N_{>q(i)}(Z_b^a)$. The remaining edges are colored blue.

Claim 2. There is no monochromatic red tight path on n vertices in this coloring.

Proof of Claim 2. Assume for a contradiction that there is a red tight path $\mathcal{R}$ on n vertices. By Claim 1, we know that the neighborhoods $N_{>q(i)}(Z_b^a)$ for $a \in [2], b \in [i]$ are pairwise disjoint. Thus, every edge in $E(\mathcal{R})$ is contained in only a single neighborhood $N_{>q(i)}(Z_b^a)$, $a \in [2], b \in [i]$. Now we distinguish two cases for $\mathcal{R}$.

Firstly, consider the case that $\mathcal{R}$ has all of its edges exclusively in one of the neighborhoods, so $E(\mathcal{R}) \subseteq N_{>q(i)}(Z_b^a)$ for some $a \in [2], b \in [i]$. Remember that the edge set Z_b^a induces a k-graph on precisely $\left\lfloor \frac{n}{2} \right\rfloor$ vertices. Then applying Corollary 5.2 for the tight path $\mathcal{R}$ and the vertex set $\cup Z_b^a$ yields a contradiction.

The other case to consider is that $E(\mathcal{R})$ contains edges in at least two of the mentioned neighborhoods. Especially, there are two edges which are consecutive in $\mathcal{R}$ and in different such sets. In other words, there exist edges $e_1 \in E(\mathcal{R}) \cap N_{>q(i)}(Z_{b_1}^{a_1})$ and $e_2 \in E(\mathcal{R}) \cap N_{>q(i)}(Z_{b_2}^{a_2})$ such that $|e_1 \cap e_2| = k-1$, where $a_j \in [2], b_j \in [i]$ and at least one of $a_1 \neq a_2$ or $b_1 \neq b_2$ holds. But this is a contradiction to Claim 1.

□

Since $\mathcal{G} \to \mathcal{P}^{(k)}_{n,k-1}$, Claim 2 implies the existence of a monochromatic blue $\mathcal{P}^{(k)}_{n,k-1}$, which we denote by $\mathcal{P}_i$. Note that each edge $e \in E(\mathcal{P}_i)$ is blue in the coloring and therefore $e \notin N_{>q(i)}(Z^a_b)$ for all $a \in [2], b \in [i]$.

By iterating this procedure for $i = 1, \ldots, \lambda$ we obtain edge sets Z^a_b for $a \in [2], b \in [\lambda]$ which are pairwise disjoint and additionally a tight path $\mathcal{P}_\lambda$ on n vertices such that each edge in $E(\mathcal{P}_\lambda)$ is not contained in any set Z^a_b. This allows for the following estimate on the number of edges in $\mathcal{G}$

$$e(\mathcal{G}) \geq \sum_{b\in[\lambda]} \left(|Z^1_b| + |Z^2_b|\right) + e(\mathcal{P}_\lambda) = \lambda(n-2k-1) + m^{(k)}_{n,k-1} \geq \lceil \log_2(k+1) \rceil \cdot m^{(k)}_{n,k-1} - k^2,$$

which concludes the proof. □

It should be highlighted that the above proof also applies to the case $k = 3$, $\ell = 2$, but does not yield an improvement of the trivial bound. Instead, a stronger bound in this case is given by Theorem 1.7.

5.3 ℓ-paths for general ℓ

In this subsection the neighborhood approach is used to prove a lower bound for the size-Ramsey number of ℓ-paths for general ℓ. Distinguishing the two cases $\ell > \frac{2}{3}k$ and $\ell \leq \frac{2}{3}k$, the proofs of Theorem 1.9 as well as Theorem 1.10 are presented.

For $\ell > \frac{2}{3}k$, we generalize the proof of Theorem 1.8 presented in the previous subsection to ℓ-paths with some minor modifications. An issue which needs to be tackled in the generalized version is that for $\ell < k-1$ it not necessarily holds that $\frac{k}{k-\ell} \in \mathbb{N}$, which is an essential argument in the proof of Corollary 5.2. As a consequence, we use the weaker version Corollary 5.3. Its weaker bound requires us in each iterative step to divide the previous chosen ℓ-path $\mathcal{P}_{i-1}$ not into two parts Z^1_i, Z^2_i on roughly $\frac{n}{2}$ vertices, but into three parts Z^1_i, Z^2_i, Z^3_i each of which has a size of about $\frac{n}{3}$.

Apart from this obstacle, the following proof is similar to the one of Theorem 1.8. Several parts which are performed analogously to the previous proof are only roughly outlined.

Proof of Theorem 1.9. Let $\mathcal{G}$ be an arbitrary k-graph with $\mathcal{G} \to \mathcal{P}^{(k)}_{n,\ell}$. We shall show that $e(\mathcal{G}) \geq \left\lceil \log_2\left(\frac{2k-\ell}{k-\ell}\right)\right\rceil \cdot m^{(k)}_{n,\ell} - 4k^2$.

We prove this bound using an iteration with $\lambda = \left\lceil \log_2\left(\frac{2k-\ell}{k-\ell}\right)\right\rceil - 1$ iteration steps. Iteratively we construct new edge-disjoint ℓ-paths using $>q$-neighborhoods.

As a parameter for the $>q$-neighborhood, let

$$q_\ell(i) = \left(1 - \frac{1}{2^i}\right) \cdot (2k-\ell) \quad \text{ for } i \in \{0, \ldots, \lambda\}.$$

Similarly to before, q_ℓ is an increasing function. Furthermore, it can be seen that $0 \leq q_\ell(i) < k$ for $i \in \{0, \dots, \lambda\}$, which provides that the $>q_\ell(i)$-neighborhood indeed exists for such i.

As an initial step, the Ramsey property $\mathcal{G} \to \mathcal{P}^{(k)}_{n,\ell}$ implies that there exists an ℓ-path $\mathcal{P}_0$ on n vertices in $\mathcal{G}$.

In the following we proceed iteratively for $i = 1, \dots, \lambda$. In each step we construct:

- Edge sets $Z_i^1, Z_i^2, Z_i^3 \subseteq E(\mathcal{P}_{i-1})$ such that each of them induces an ℓ-path in $\mathcal{G}$ on roughly $\frac{n-k}{3}$ vertices which is vertex-disjoint from the other two such ℓ-paths.
- An ℓ-path $\mathcal{P}_i$ on n vertices such that $E(\mathcal{P}_i) \cap N_{>q(i)}(Z_b^a) = \varnothing$ for all $a \in [3], b \in [i]$.

Firstly, we subdivide $\mathcal{P}_{i-1}$ into three pairwise vertex-disjoint ℓ-paths of roughly the same size. Note that such a construction requires more care than in the case of tight paths. In a tight path, a set of at least k consecutive path vertices always induces tight subpath. For $\ell \neq k-1$, such a set possibly induces a subhypergraph with isolated vertices, i.e. not an ℓ-path.

Consider a path enumeration of the vertices $V(\mathcal{P}_{i-1}) = \{v_1, \dots, v_n\}$. Now let $n' \in \mathbb{N}$ be the unique natural number such that

$$\frac{n-k}{3} - k + \ell < n' \leq \frac{n-k}{3} \quad \text{and} \quad \frac{n'-\ell}{k-\ell} \in \mathbb{N}.$$

Additionally, let $j \in \mathbb{N}, 0 \leq j < k - \ell$ be the unique natural number such that $\frac{n'+j}{k-\ell} \in \mathbb{N}$. Then consider the vertex sets

$$V_i^1 = \Big\{v_1, \dots, v_{n'}\Big\}, V_i^2 = \Big\{v_{n'+j+1}, \dots, v_{2n'+j}\Big\} \text{ and } V_i^3 = \Big\{v_{2n'+2j+1}, \dots, v_{3n'+2j}\Big\}.$$

The choice of n' and j provides that each $\mathcal{P}_{i-1}[V_i^a]$ is indeed a well-defined ℓ-path. (Note that the edges in $\mathcal{P}_{i-1}$ have the form $\{v_{\rho(k-\ell)+1}, \dots, v_{\rho(k-\ell)+k}\}$ for $\rho \in \{0, \dots, m^{(k)}_{n,\ell} - 1\}$ and that $3n' + 2j \leq n - k + 2(k-\ell) \leq n - \frac{k}{3} \leq n$.)

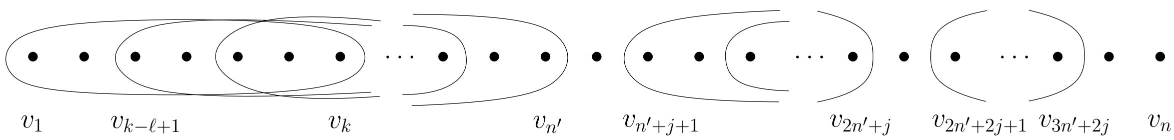

Figure 7: Enumerated vertices of $\mathcal{P}_{i-1}$ with associated edges for $k = 7, \ell = 5$

Now let $Z_i^a = E(\mathcal{P}_{i-1}[V_i^a])$ for $a \in [3]$. It can be seen that

$$|Z_i^a| = \frac{n'-\ell}{k-\ell} \geq \frac{n-4k}{3(k-\ell)}.$$

Regarding the already defined edge sets Z_b^a, $a \in [3], b \in [i]$ we show the following claim.

Claim. Let $a_1, a_2 \in [3]$, $b_1, b_2 \in [i]$ such that at least one of $a_1 \neq a_2$ or $b_1 \neq b_2$ holds (so excluding the case $a_1 = a_2 \wedge b_1 = b_2$). Then for each two edges $e_1 \in N_{>q_\ell(i)}(Z_{b_1}^{a_1})$ and $e_2 \in N_{>q_\ell(i)}(Z_{b_2}^{a_2})$ we have

$$|e_1 \cap e_2| < \ell.$$

The claim is verified analogously to Claim 1 in the previous proof.

In a next step consider the following coloring of $\mathcal{G}$. Assign the color red to each edge in $N_{>q_\ell(i)}(Z_b^a)$ for all $a \in [3], b \in [i]$. All other edges are colored blue. As in the proof of Theorem 1.8, it can be seen that in this coloring there is no monochromatic red ℓ-path on n vertices (using Corollary 5.3 instead of Corollary 5.2 and the above claim instead of Claim 1 of the mentioned proof). By the Ramsey property $\mathcal{G} \to \mathcal{P}_{n,\ell}^{(k)}$, we know that there is a monochromatic blue such ℓ-path in $\mathcal{G}$, which we denote by $\mathcal{P}_i$. Observe that for each edge $e \in E(\mathcal{P}_i)$, it holds that $e \notin N_{>q(i)}(Z_b^a)$ for all $a \in [3], b \in [i]$.

We iterate this procedure for $i = 1, \ldots, \lambda$. After the λ-th step we obtain edge sets Z_b^a for $a \in [3], b \in [\lambda]$ and $E(\mathcal{P}_\lambda)$ which are pairwise disjoint. Consequently, the number of edges in $\mathcal{G}$ is at least

$$\begin{aligned} e(\mathcal{G}) &\geq \sum_{b \in [\lambda]} \Big(|Z_b^1| + |Z_b^2| + |Z_b^3| \Big) + e(\mathcal{P}_\lambda) \geq \lambda \cdot \frac{n-4k}{k-\ell} + m_{n,\ell}^{(k)} \\ &= \Big(\lambda + 1\Big) m_{n,\ell}^{(k)} - \frac{\lambda(4k-\ell)}{k-\ell} > \left\lceil \log_2 \left(\frac{2k-\ell}{k-\ell} \right) \right\rceil \cdot m_{n,\ell}^{(k)} - 4k^2. \end{aligned}$$

□

Note that the proof of Theorem 1.9 can be executed analogously for $\frac{k+1}{2} < \ell \leq \frac{2}{3}k$ (Corollary 5.3 also holds for such ℓ), but yields no improvement on the trivial bound. However, with a slight twist the neighborhood approach provides a refined bound, which is presented in the following, utilizing the $\geq q$-neighborhood. For this proof we only construct three pairwise edge-disjoint ℓ-paths. Although we perform similar arguments as before, the following proof is not formulated iteratively but instead the individual steps are stated explicitly.

Proof of Theorem 1.10. If $\ell \in \{\frac{k}{2}, \frac{2}{3}k\}$, let $c(k,\ell) = \frac{\ell}{k}$, otherwise let $c(k,\ell) = \frac{\ell}{2k-\ell-1}$. Let $\mathcal{G}$ be an arbitrary k-uniform hypergraph which has the Ramsey property $\mathcal{G} \to \mathcal{P}_{n,\ell}^{(k)}$. As before, we show that $\mathcal{G}$ is a k-graph on at least $\Big(2 + c(k,\ell)\Big) \cdot m_{n,\ell}^{(k)} - 4$ many edges.

It is obvious that there exists some ℓ-path $\mathcal{P}_0$ on n vertices in $\mathcal{G}$. We restrict our view to a subhypergraph of $\mathcal{P}_0$. Let n' be the unique natural number with

$$c(k,\ell)\Big(n-\ell\Big)-k+\ell \le n' < c(k,\ell)\Big(n-\ell\Big) \quad \text{and} \quad \frac{n'-\ell}{k-\ell}\in\mathbb{N}.$$

Consider a path enumeration of the vertices $V(\mathcal{P}_0)=\{v_1,\dots,v_n\}$. Now let

$$\mathcal{P}_0'=\mathcal{P}_0\Big[\{v_1,\dots,v_{n'}\}\Big].$$

By the choice of n' it can be seen that $\mathcal{P}_0'$ is a well-defined ℓ-path on n' vertices. Observe that

$$e(\mathcal{P}_0') > \frac{c(k,\ell)(n-\ell)-k}{k-\ell} \ge c(k,\ell)m_{n,\ell}^{(k)}-3,$$

using that $\ell \le \frac{2}{3}k$.

In order to find an ℓ-path $\mathcal{P}_1$ which is edge-disjoint from $\mathcal{P}_0'$, we proceed similarly to the proof of Theorem 1.8 given in the previous subsection by considering the following coloring. Color all edges in the $\ge\ell$-neighborhood $N_{\ge\ell}\Big(E(\mathcal{P}_0')\Big)$ in red and the remaining edges in blue. Assume for a contradiction that there is a monochromatic red $\mathcal{P}_{n,\ell}^{(k)}$, say $\mathcal{R}$, in this coloring. Then Proposition 5.1 applied on the ℓ-path $\mathcal{R}$ and the vertex set $V(\mathcal{P}_0')$ provides a contradiction.

Since $\mathcal{G}\to\mathcal{P}_{n,\ell}^{(k)}$, there is a monochromatic blue ℓ-path on n vertices in $\mathcal{G}$. Especially, there is also a blue ℓ-path on $m_{n,\ell}^{(k)}-1$ many edges, i.e. an ℓ-path on one edge less, contained in this structure. Fix such an ℓ-path $\mathcal{P}_1'$ with $e(\mathcal{P}_1')=m_{n,\ell}^{(k)}-1$. Then note that $N_{\ge\ell}\Big(E(\mathcal{P}_0')\Big)$ and $E(\mathcal{P}_1')$ are disjoint edge sets.

In order to find a third edge-disjoint ℓ-path, we consider another coloring in the following. Now we assign the color red to each edge in $E(\mathcal{P}_0')\cup E(\mathcal{P}_1')$ and the color blue to all other edges. Assume for a contradiction that there is a red ℓ-path $\mathcal{R}$ on n vertices in this coloring. Then neither $E(\mathcal{R})\subseteq E(\mathcal{P}_0')$ nor $E(\mathcal{R})\subseteq E(\mathcal{P}_1')$ since the two edge sets have a size strictly less than $m_{n,\ell}^{(k)}$.

Therefore, $\mathcal{R}$ consists of edges of both $E(\mathcal{P}_0')$ and $E(\mathcal{P}_1')$. By definition, each red edge of this coloring is either in $E(\mathcal{P}_0')$ or in $E(\mathcal{P}_1')$, so there exist two edges $e_1\in E(\mathcal{P}_0')\cap E(\mathcal{R}), e_2\in E(\mathcal{P}_1')\cap E(\mathcal{R})$ which are consecutive in $\mathcal{R}$, i.e. $|e_1\cap e_2|=\ell$. But that is a contradiction to the fact that $N_{\ge\ell}\Big(E(\mathcal{P}_0')\Big)$ and $E(\mathcal{P}_1')$ are disjoint.

Consequently, there is no monochromatic red $\mathcal{P}_{n,\ell}^{(k)}$ in this coloring. By the same argument as before, there is a blue ℓ-path $\mathcal{P}_2$ on n vertices in the k-graph $\mathcal{G}$. Then the three edge sets $E(\mathcal{P}_0'), E(\mathcal{P}_1'), E(\mathcal{P}_2)$ are pairwise disjoint. Now we obtain an estimate on the number of edges in $\mathcal{G}$,

$$e(\mathcal{G})\ge e(\mathcal{P}_0')+e(\mathcal{P}_1')+e(\mathcal{P}_2)\ge\Big(2+c(k,\ell)\Big)\cdot m_{n,\ell}^{(k)}-4.$$

□

It is worth mentioning that the bottleneck for this proof is Proposition 5.1. Proceeding more carefully with the estimates in this lemma provides the Ramsey bound

$$\hat{R}^{(k)}(\mathcal{P}_{n,\ell}^{(k)}) \geq \left(2 + \frac{\ell}{k+\rho}\right) \cdot m_{n,\ell}^{(k)} - 4,$$

where $\rho \in \mathbb{N}$ is the smallest natural number with $\frac{k+\rho}{k-\ell} \in \mathbb{N}$. (Note that this yields no improvement for $\ell = 1$, $\ell = \frac{k}{2}$ nor $\ell = \frac{2}{3}k$.)

Theorem 1.10 also implies a bound for the size-Ramsey number of loose paths. For $k \geq 3$ and $n \in \mathbb{N}$ sufficiently large such that $m_{n,1}^{(k)} \in \mathbb{N}$, we have shown the bound

$$\hat{R}^{(k)}(\mathcal{P}_{n,1}^{(k)}) \geq \left(2 + \frac{1}{2k-2}\right) \cdot m_{n,1}^{(k)} - 4,$$

which especially proves Theorem 1.5. In addition to that, the above result implies Theorem 1.7 (using $k = 3$, $\ell = 2$) and Corollary 1.11 (for $k = 2$, $\ell = 1$).

6 Concluding remarks

In this thesis lower bounds for the size-Ramsey number of k-uniform ℓ-paths have been presented for all $1 \leq \ell \leq k-1$. It is worth noting that all stated lower bounds have the same form

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell}) \geq C \cdot m^{(k)}_{n,\ell} \quad \text{for sufficiently large } n \text{ with } m^{(k)}_{n,\ell} \in \mathbb{N},$$

where the leading factor $C > 0$ is independent from n. We compare the different results for $k \geq 2$ and $1 \leq \ell \leq k-1$ by considering

$$C(k,\ell) = \liminf_{n\to\infty} \frac{\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,\ell})}{m^{(k)}_{n,\ell}},$$

where we only take those n into account for which $m^{(k)}_{n,\ell} \in \mathbb{N}$. By the trivial lower bound (see Observation 2.9) it is obvious that $C(k,\ell) \geq 2$ for all k and ℓ.

In the loose case $\ell = 1$, it remains open in this thesis to find a notable improvement of this trivial bound when considering a large uniformity k, with Theorem 1.6 only providing a leading factor

$$C(k,1) \geq 2 + \frac{1}{k+3} \to 2 \quad \text{for } k \to \infty.$$

As mentioned with the proof, most of the estimates used for showing the result enable a bound $2 + \varepsilon$ for some $\varepsilon > 0$ independent from k. However, it stays unclear how to overcome the bottleneck of the proof.

Additionally, the upper bound in the loose case given in Proposition 2.3 appears to be very rough. The mentioned k-graph $\mathcal{G}$ witnessing the stated bound has the property that for large k the vast majority of vertices contained in each edge have degree 1 in $\mathcal{G}$, so are contained in no other edge of the k-graph. This seems to be a suboptimal choice since it largely simplifies the construction of a proper edge 2-coloring of $\mathcal{G}$. Those observations motivate the following conjecture.

Conjecture 6.1. *Let $\varepsilon > 0$. Then for $k \geq 2$ and $n \in \mathbb{N}$ both sufficiently large with $m^{(k)}_{n,1} \in \mathbb{N}$,*

$$\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,1}) \leq \Big(2 + \varepsilon\Big) \cdot m^{(k)}_{n,1}.$$

For ℓ linear in terms of k, i.e. $\ell = \alpha k$ for some fixed constant $\alpha \in \mathbb{Q}$, $0 < \alpha < 1$, Theorem 1.10 and Theorem 1.9 (as well as Theorem 4.2 for $\ell \leq \frac{k}{2}$) yield a notable refinement of the trivial bound. Reformulating these results, we obtain for $k \geq 2$ and $\ell = \alpha k \in \mathbb{N}$ where $0 < \alpha < 1$,

$$C(k,\alpha k) \geq 2 + \frac{\alpha}{2-\alpha}.$$

For such linear ℓ, a comparison of the hypergraph and graph case is especially interesting. Using the notation of $C(k, \ell)$, Observation 2.10 provides that

$$C(k, \ell) \leq C(2, 1), \qquad \text{for } \ell \leq \tfrac{k}{2},$$

where $C(2, 1)$ describes the leading factor in the graph case. For $\ell = \alpha k$ with α close to 1, a similar statement is disproven by the results of this thesis. Combining the lower bound Theorem 1.9 with the upper bound Theorem 2.1 we obtain that

$$C(k, \ell) > C(2, 1), \qquad \text{if } \ell > \left(1 - 2^{-74}\right)k,$$

a statement which possibly also holds for other $\ell > \frac{k}{2}$.

Conjecture 6.2. *Let $k \geq 3$ and $\ell > \frac{k}{2}$. Then*

$$C(k, \ell) > C(2, 1).$$

In the tight case $\ell = k - 1$, Theorem 1.8 provides the leading factor

$$C(k, k-1) = \liminf_{n \to \infty} \frac{\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,k-1})}{n} \geq \left\lceil \log_2(k+1) \right\rceil \to \infty \quad \text{for } k \to \infty.$$

Since $C(2, 1) \leq 74$ by Theorem 2.1, this especially proves that $\hat{R}^{(k)}(\mathcal{P}^{(k)}_{n,k-1}) \geq \hat{R}(P_n)$ for sufficiently large n and k. With a view to Conjecture 2.6 this result can be seen as an early step towards a potential disproof of the conjecture. On the other hand, given the case that the mentioned conjecture is proven, the analysis of the leading factor $C(k, k-1)$ asymptotically in terms of k remains a question to consider.

Besides the results on the size-Ramsey number of k-uniform ℓ-paths, the presented proof concepts also apply to the graph case. Noteworthy, Corollary 1.11 reproves the lower bound for $\hat{R}(P_n)$ given by Dudek and Pralat [14]. Although not achieving the quality of the best known bound by Bal and DeBiasio [3], it is worth noting that the proof stated here avoids the arguments described in Section 3 (vertex partitions, spanning trees or minimum degree arguments) and instead concentrates on neighborhoods of edge sets. Elaborating such an approach – optimized for the graph case – might be useful for improving the current best known lower bound.

7 Appendix

In the following we prove Theorem 4.3 using the crossing approach and the related notation introduced in Subsection 4.1.

Proof of Theorem 4.3. Let $\ell = \frac{k}{2}$ and consider an arbitrary k-graph $\mathcal{G}$ with

$$e(\mathcal{G}) = \left\lfloor \tfrac{9}{4} m_{n,\ell}^{(k)} \right\rfloor - 6.$$

We show that there is a coloring of $\mathcal{G}$ which does not contain a $\mathcal{P}_{n,\ell}^{(k)}$. More precisely, we construct a bad coloring, so a coloring of $\mathcal{G}$ with precisely $m_{n,\ell}^{(k)} - 1$ many blue edges which contains no red ℓ-path on n vertices. In this proof a bad coloring has precisely $\left\lfloor \frac{5}{4} m_{n,\ell}^{(k)} \right\rfloor - 5$ many red edges. By Observation 4.1 we suppose that there are two edge-disjoint ℓ-paths $\mathcal{P}_1, \mathcal{P}_2$ on $m_{n,\ell}^{(k)} - 1$ edges each. Note that there are exactly $\left\lfloor \frac{1}{4} m_{n,\ell}^{(k)} \right\rfloor - 4$ additional edges in $\mathcal{G}$.

We introduce some further notation for segments of ℓ-paths, i.e. 2-sets of consecutive path edges. Given an ℓ-path $\mathcal{P}$ and a segment $S = \{e_1, e_2\}$ of $\mathcal{P}$, we say that the *segment intersection* $I(S)$ of S is the vertex set $e_1 \cap e_2$. Note that $|I(S)| = \frac{k}{2}$ as $\mathcal{P}$ is an ℓ-path.

In this proof we restrict our view to crossings of $\mathcal{P}_1$ and $\mathcal{P}_2$ with two overlapping edges (i.e. two edges in $E(\mathcal{P}_2)$). Such crossings have the following characterization using segments. A crossing with two overlapping edges is an ordered pair of non-empty edge sets (E_1, E_2) such that

$$E_2 \text{ is a segment } S \text{ of } \mathcal{P}_2 \quad \text{and} \quad E_1 = \Big\{ e \in E(\mathcal{P}_1) \colon I(S) \subseteq e \Big\}.$$

Observe that $I(S)$ is the unique crossing vertex set of such a crossing (E_1, E_2).

It can be seen that crossings with two overlapping edges are either marginal (if there is only one base edge and it is an end edge of $\mathcal{P}_1$), $(1,2)$-structured or $(2,2)$-structured. In the following we focus our view on non-marginal crossings and in order to improve readability we refer to $(2,2)$-structured crossings as *symmetric crossings* and to $(1,2)$-structured crossings as *crooked crossings* (see the example below).

Let $B_{\text{sym}} \subseteq E(\mathcal{P}_1)$ be the set consisting of all base edges of symmetric crossings in $\mathcal{G}$ and let $\rho_{\text{sym}} \in \mathbb{N}$ be the maximum number of pairwise base-disjoint such crossings. Similarly, let B_{cro} be the set of edges in $E(\mathcal{P}_1)$ which are the base edge of a crooked crossing and $\rho_{\text{cro}} \in \mathbb{N}$ be the maximum number of pairwise base-disjoint crooked crossings. Note that $\rho_{\text{cro}} = |B_{\text{cro}}|$.

Example.
Consider two 4-uniform 2-paths $\mathcal{P}_1$ and $\mathcal{P}_2$ which intersect as illustrated below. Then $(\{e_{1,2}, e_{1,3}\}, \{e_{2,2}, e_{2,3}\})$ is a symmetric and $(\{e_{1,4}\}, \{e_{2,4}, e_{2,5}\})$ is a crooked crossing.

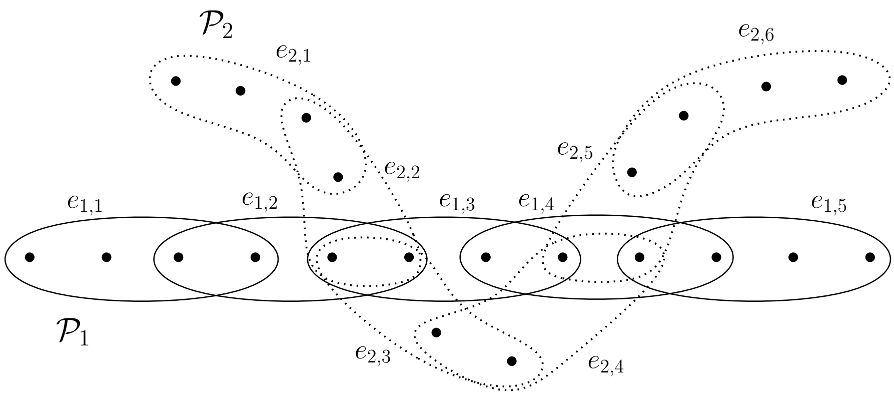

Figure 8: Example of a symmetric and a crooked crossing

Claim 1. The sets B_{sym} and B_{cro} are disjoint.

Proof of Claim 1. Assume for a contradiction that there is an edge $e \in B_{\text{sym}} \cap B_{\text{cro}}$. Then e is the base edge of a (non-marginal) crooked crossing and therefore not an end edge of $\mathcal{P}_1$. By this reason there exist two distinct edges $e_1, e_2 \in E(\mathcal{P}_1)$ with $|e \cap e_1| = \frac{k}{2}, |e \cap e_2| = \frac{k}{2}$ and $e = (e \cap e_1) \cup (e \cap e_2)$. Since $e \in B_{\text{sym}}$, there is a symmetric crossing such that without loss of generality $\{e, e_1\}$ is its base edge set. Especially there exists a segment S in $\mathcal{P}_2$ such that $I(S) = e \cap e_1$.

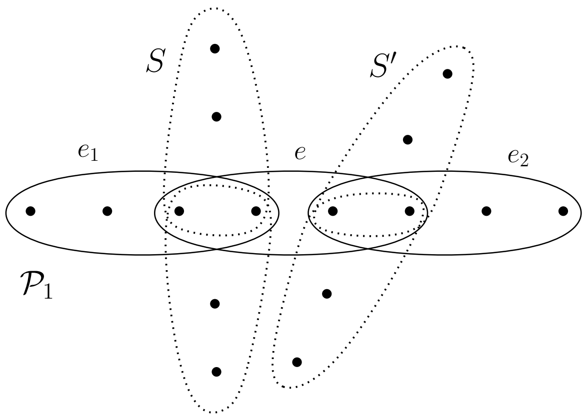

Figure 9: Constellation of the segments S and S' in the case $k = 4$

By the assumption that $e \in B_{\text{cro}}$, we additionally find a segment S' in $\mathcal{P}_2$ such that $(\{e\}, S')$ is a crooked crossing. Note that $I(S') \subseteq e$ and furthermore $I(S') \neq I(S)$ because these segment intersections are the crossing vertex sets of distinct crossings, so especially $S \neq S'$. Note that the ℓ-path $\mathcal{P}_2$ has maximum degree 2, therefore we obtain that $I(S') \cap I(S) = \varnothing$, so $I(S') = e \backslash I(S) = e \cap e_2$. The above mentioned crossing $(\{e\}, S')$ has this set $I(S')$ as its unique crossing vertex set. Now $I(S') \subseteq e_2$ and so e_2 is a second base edge of this crossing, a contradiction. □

In the following we distinguish between two cases and construct a bad coloring for each of them.

Case 1. It holds that $\rho_{\text{sym}} + \frac{1}{2}\rho_{\text{cro}} \geq \frac{1}{4}m_{n,\ell}^{(k)}$.

In this case let M_{sym} be a set of ρ_{sym} many pairwise base-disjoint symmetric crossings and similarly let M_{cro} be a set of consisting of ρ_{cro} many pairwise base-disjoint crooked crossings. We consider $M = M_{\text{sym}} \cup M_{\text{cro}}$. By Claim 1, the crossings in M are pairwise base-disjoint.

In a next step we will find a large subset of M whose elements are pairwise edge-disjoint. Remember that for every element of M the overlapping edge set is a segment of $\mathcal{P}_2$. As introduced in Subsection 1.1, there is a path enumeration of the segments of $\mathcal{P}_2$. Let M^1 be the set consisting of all crossings in M which have an overlapping edge with an odd label in this enumeration and let $M^2 = M \backslash M^1$, so those crossings with an evenly labelled overlapping edge set.

It can be seen that for a fixed $i \in [2]$ the crossings in M^i are pairwise edge-disjoint. For $i \in [2]$, let ρ_{sym}^i and ρ_{cro}^i be the number of symmetric and crooked crossings in M^i, respectively. Then clearly $\rho_{\text{sym}} = \rho_{\text{sym}}^1 + \rho_{\text{sym}}^2$ and $\rho_{\text{cro}} = \rho_{\text{cro}}^1 + \rho_{\text{cro}}^2$. Without loss of generality, we suppose that

$$\rho_{\text{sym}}^1 + \tfrac{1}{2}\rho_{\text{cro}}^1 \geq \tfrac{1}{8}m_{n,\ell}^{(k)}.$$

In a next step pick a subset $M' \subset M_1$ of symmetric and crooked crossings such that

$$\tfrac{1}{8}m_{n,\ell}^{(k)} \leq \rho'_{\text{sym}} + \tfrac{1}{2}\rho'_{\text{cro}} < \tfrac{1}{8}m_{n,\ell}^{(k)} + 1,$$

where ρ'_{sym} and ρ'_{cro} are the number of symmetric and crooked crossings in M', respectively.

Now we construct a bad coloring using this set M'. Firstly, assign the color red to each edge corresponding to a crossing in M', so each base and overlapping edge. Note that the number of such edges is at most

$$4\rho'_{\text{sym}} + 3\rho'_{\text{cro}} \leq 6\left(\rho'_{\text{sym}} + \tfrac{1}{2}\rho'_{\text{cro}}\right) \leq \tfrac{3}{4}m_{n,\ell}^{(k)} + 6.$$

Furthermore, color the remaining edges of $\mathcal{G}$ arbitrarily in red or blue so that there are precisely $m_{n,\ell}^{(k)} - 1$ blue edges. (This is possible since $\frac{3}{4}m_{n,\ell}^{(k)} + 6 \leq \left\lfloor \frac{5}{4}m_{n,\ell}^{(k)} \right\rfloor - 5$.)

In order to complete Case 1, it remains to show that this coloring is indeed bad, i.e. that there is no red $\mathcal{P}_{n,\ell}^{(k)}$ in $\mathcal{G}$. So let $\mathcal{R}$ be a monochromatic red k-uniform ℓ-path of maximal size. The maximum degree of $\mathcal{R}$ is 2, so we know that for each crossing in M' only at most two of its corresponding edge are in $E(\mathcal{R})$. In other words, for each symmetric crossing, there are at least two red edges not contained in $E(\mathcal{R})$ and for each crooked crossing at least one edge. Consequently,

$$e(\mathcal{R}) \leq \left(\left\lfloor \tfrac{5}{4} m_{n,\ell}^{(k)} \right\rfloor - 5\right) - 2\rho'_{\text{sym}} - \rho'_{\text{cro}} \leq \left(\tfrac{5}{4} m_{n,\ell}^{(k)} - 5\right) - \tfrac{2}{8} m_{n,\ell}^{(k)} \leq m_{n,\ell}^{(k)} - 5.$$

This implies that the coloring is bad.

Case 2. It holds that $\rho_{\text{sym}} + \frac{1}{2}\rho_{\text{cro}} < \frac{1}{4} m_{n,\ell}^{(k)}$.

Analogously to the claim in the proof of Theorem 1.6 (see Subsection 4.3), observe that the number of edges in $E(\mathcal{P}_1)$ which are the base edge of a symmetric crossing is at most

$$3\rho_{\text{sym}} < \tfrac{3}{4} m_{n,\ell}^{(k)} - \tfrac{3}{2}\rho_{\text{cro}}.$$

Once more, we construct a bad coloring. Color the following edges in blue:

(1) Each base edge of a symmetric crossing

(2) Each base edge of a crooked crossing

(3) Each additional edge

(4) Each of the end edges of the ℓ-paths $\mathcal{P}_1$ and $\mathcal{P}_2$

The number of all mentioned edges is bounded by

$$3\rho_{\text{sym}} + \rho_{\text{cro}} + \left(\left\lfloor \tfrac{1}{4} m_{n,\ell}^{(k)} \right\rfloor - 4\right) + 4 < \tfrac{3}{4} m_{n,\ell}^{(k)} - \tfrac{3}{2}\rho_{\text{cro}} + \rho_{\text{cro}} + \tfrac{1}{4} m_{n,\ell}^{(k)} \leq m_{n,\ell}^{(k)}.$$

Since $m_{n,\ell}^{(k)} \in \mathbb{N}$, it equivalently holds that there are at most $m_{n,\ell}^{(k)} - 1$ such edges. Color the remaining uncolored edges arbitrarily such that there are exactly $m_{n,\ell}^{(k)} - 1$ blue edges.

We shall show that there is no monochromatic red $\mathcal{P}_{n,\ell}^{(k)}$. Assume for a contradiction that there actually exists a red such ℓ-path, denoted by $\mathcal{R}$. Clearly, $E(\mathcal{R}) \subseteq E(\mathcal{P}_1) \cup E(\mathcal{P}_2)$ since every additional edge is colored blue. We proceed by carefully examining the occurrences of edges of the two ℓ-paths in $E(\mathcal{R})$.

A segment in $\mathcal{R}$ is said to be *switching* if it consists of one edge in $E(\mathcal{P}_1)$ and one edge in $E(\mathcal{P}_2)$. Furthermore, we say that a segment S in $\mathcal{R}$ is 2-*parallel* if there is a segment T in $\mathcal{P}_2$ such that its segment intersection is $I(T) = I(S)$. Note that both $S = T$ and $S \neq T$ is possible.

Claim 2. There is no 2-parallel segment of the ℓ-path $\mathcal{R}$.

Proof of Claim 2. Assume that there is a 2-parallel segment S of $\mathcal{R}$, and let T be the segment in $\mathcal{P}_2$ with $I(T) = I(S)$. Obviously, $S \subseteq E(\mathcal{P}_1) \cup E(\mathcal{P}_2)$. If $S \neq T$, then there is an edge $e \in S \cap E(\mathcal{P}_1)$. This edge $e \in E(\mathcal{P}_1)$ and the edges in $T \subseteq E(\mathcal{P}_2)$ are contained in a crossing using the crossing vertex set $I(T)$ (since $I(T) = I(S) \subseteq e$). Especially, e is either an end edge of $\mathcal{P}_1$ or a base edge of symmetric or crooked crossing. In each case, e is colored blue, which is a contradiction to $e \in E(\mathcal{R})$. Thus $S = T$, so S is also a segment in $\mathcal{P}_2$.

Now consider an arbitrary segment of $\mathcal{R}$ consecutive to S, i.e. let S' be a segment in $\mathcal{R}$ such that there is an edge $e' \in E(\mathcal{R})$ with $e' \in S \cap S'$. As seen in the last paragraph, $e' \in S$ implies $e' \in E(\mathcal{P}_2)$ and using the fact that e' is a red edge we obtain that e' is not an end edge of $\mathcal{P}_2$. Hence, there is a segment T' in $\mathcal{P}_2$ with $S \neq T'$ and which contains e'. Now observe that $I(S') = e' \backslash I(S) = I(T')$ because $\mathcal{R}$ and $\mathcal{P}_2$ are both $\frac{k}{2}$-paths. This implies that the segment S' is also 2-parallel, witnessed by T'. Then, by the same argument as before, S also only consists of edges from $E(\mathcal{P}_2)$.

Successively, it can be seen that each segment of $\mathcal{R}$ is 2-parallel and $E(\mathcal{R}) \subseteq E(\mathcal{P}_2)$. But $e(\mathcal{P}_2) \leq m_{n,\ell}^{(k)} - 1$, which is a contradiction to the fact that $\mathcal{R}$ is an ℓ-path on n vertices. $\square$

In the following we use this claim to find a contradiction to the existence of $\mathcal{R}$. For this purpose, we provide a relation between the edge numbers $e(\mathcal{R})$ and $e(\mathcal{P}_1)$ by considering switching segments of $\mathcal{R}$. First of all, let $r_1 = |E(\mathcal{R}) \cap E(\mathcal{P}_1)|$ and $r_2 = |E(\mathcal{R}) \cap E(\mathcal{P}_2)|$. Then

$$r_1 + r_2 = e(\mathcal{R}) = m_{n,\ell}^{(k)}.$$

Consider the edges in $E(\mathcal{R}) \cap E(\mathcal{P}_2)$. If two edges in this intersection have $\frac{k}{2}$ many vertices in common, they form a 2-parallel segment of $\mathcal{R}$, a contradiction to Claim 2. Consequently, every segment of $\mathcal{R}$ containing an edge in $E(\mathcal{P}_2)$ is switching, i.e. that the second edge of the segment is in $E(\mathcal{P}_1)$.

Now consider a fixed switching segment $\{e_1, e_2\}$ of $\mathcal{R}$ where $e_1 \in E(\mathcal{P}_1)$ and $e_2 \in E(\mathcal{P}_2)$. Since $\mathcal{R}$ is a k-graph with maximum vertex degree 2, each edge in $E(\mathcal{G}) \backslash \{e_1, e_2\}$ containing a vertex in $e_1 \cap e_2$ is not an element of $E(\mathcal{R})$. We say that such edges are *blocked* by the switching segment $\{e_1, e_2\}$. In the following we focus our view to blocked edges in $E(\mathcal{P}_1)$, so let E' be the set of all edges in $E(\mathcal{P}_1)$ which are blocked by any switching segment. Then the edge sets $E(\mathcal{R}) \cap E(\mathcal{P}_1)$ and E' are disjoint subsets of $E(\mathcal{P}_1)$ and especially

$$e(\mathcal{P}_1) \geq r_1 + |E'|.$$

Claim 3. $|E'| \geq r_2$

Proof of Claim 3. Let $s \in \mathbb{N}$ be the amount of switching segments in $\mathcal{R}$. We prove the inequality by bounding s in terms of r_2 as well as in terms of $|E'|$. Let ρ denote the number of end edges of $\mathcal{R}$ which are in $E(\mathcal{P}_2)$, then clearly $0 \leq \rho \leq 2$.

For the first part, remember that every segment in $\mathcal{R}$ which contains an edge in $E(\mathcal{P}_2)$ is switching and each edge in $E(\mathcal{R})$ is element of precisely two segments of $\mathcal{R}$ unless it is an end edge of the ℓ-path. This provides

$$s = 2r_2 - \rho.$$

Now we find a relation of s and $|E'|$. Let $\{e_1, e_2\}$ be an arbitrary switching segment of $\mathcal{R}$ where $e_1 \in E(\mathcal{P}_1)$ and $e_2 \in E(\mathcal{P}_2)$. We know that $e_1 \in E(\mathcal{R})$ is colored in red and therefore e_1 is not an end edge of $\mathcal{P}_1$. Hence, there are precisely two edges in $E(\mathcal{P}_1)$ which share $\frac{k}{2}$ many vertices with e_1. At the same time, there is at most one edge in $E(\mathcal{R})\backslash\{e_2\}$ which has $\frac{k}{2}$ vertices with e_1 in common. So at least one of the edges which is consecutive to e_1 in $\mathcal{P}_1$ is blocked by the segment $\{e_1, e_2\}$. Consequently, each switching segment of $\mathcal{R}$ blocks at least one edge in $E(\mathcal{P}_1)$.

It remains to examine by how many segments each edge in E' is blocked. A rough bound for this is easy to obtain: If an edge $e \in E(\mathcal{P}_1)$ is blocked by some switching segment, then especially the switching segment contains an edge in $E(\mathcal{P}_1)$ which has a vertex in common with e. Since $\mathcal{P}_1$ is an ℓ-path, there are only at most two candidates for such an edge, the edges consecutive to e in $\mathcal{P}_1$. This implies that each edge in E' is blocked by at most two switching segments, which provides the rough estimate $|E'| \geq \frac{1}{2}s = r_2 - \frac{1}{2}\rho$. This does not suffice to verify the claim, so it is required to proceed more carefully in order to obtain a refined inequation.

We want to show that there are at least ρ many edges in E' which are solely blocked by one switching segment. For this purpose, consider a path enumeration of the edges of $\mathcal{P}_1$ (as introduced in Subsection 1.1). Especially, every edge in $E(\mathcal{P}_1) \cap E(\mathcal{R})$ and in E' is enumerated by this labeling.

Let $e \in \Big(E(\mathcal{R}) \cap E(\mathcal{P}_1)\Big) \cup E'$ be the edge with the lowest enumeration label.

If $e \in E'$, it is easy to see that this edge is only blocked by one switching segment since the predecessor of e in $\mathcal{P}_1$ is clearly not in $\mathcal{R}$.

So suppose that $e \in E(\mathcal{R}) \cap E(\mathcal{P}_1)$. We show that in this case e is an end edge of $\mathcal{R}$. As before, $e \in E(\mathcal{R})$ implies that e is not an end edge of $\mathcal{P}_1$. Thus there is an edge $e' \in E(\mathcal{P}_1)$ which is the predecessor of e in the mentioned path enumeration of $E(\mathcal{P}_1)$ (and so $|e \cap e'| = \frac{k}{2}$). By the definition of e, it holds that e' is neither in $E(\mathcal{R})$ nor in E'.

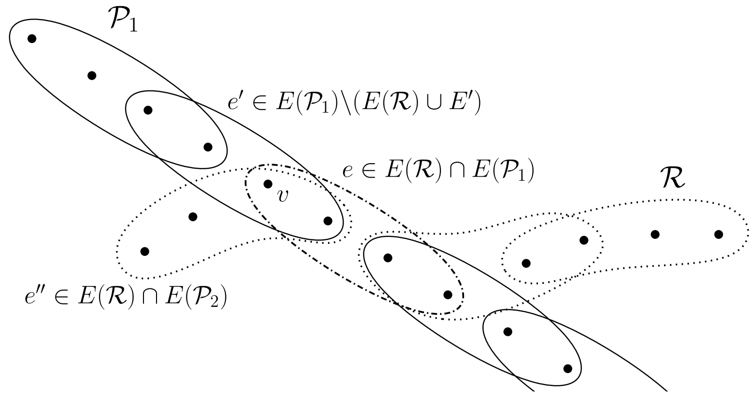

Figure 10: Constellation of e, e' and e'' which yields a contradiction

Now assume that e is not an end edge of $\mathcal{R}$. Then fix a vertex $v \in e \cap e'$ and note that v has degree 2 in $\mathcal{R}$ (otherwise e is an end edge). Since $e' \notin E(\mathcal{R})$, there is an edge $e'' \in E(\mathcal{R}) \cap E(\mathcal{P}_2)$ with $v \in e''$. Then e and e'' form a switching segment which blocks e' witnessed by the vertex v. This is a contradiction to $e' \notin E'$.

Consequently, either the lowest enumerated edge in E' is solely blocked by a single switching segment of $\mathcal{R}$; or the edge in $E(\mathcal{R}) \cap E(\mathcal{P}_1)$ with the lowest enumeration label is an end edge of the ℓ-path $\mathcal{R}$. An analogous observation holds regarding the highest enumerated edge.

This yields that for each edge counted by ρ there is a distinct element of E' which is blocked by only one switching segment. Consequently, we observe that $|E'| - \rho \geq \frac{1}{2}(s - \rho)$ and so

$$|E'| \geq \frac{1}{2}(s + \rho) \geq r_2.$$

□

Combining the above stated bounds, we obtain

$$e(\mathcal{P}_1) \geq r_1 + |E'| \geq r_1 + r_2 = e(\mathcal{R}).$$

This is a contradiction since $e(\mathcal{P}_1) = m_{n,\ell}^{(k)} - 1$, but $e(\mathcal{R}) = m_{n,\ell}^{(k)}$. □

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Erklärung

Die vorliegende Arbeit habe ich selbständig verfasst und keine anderen als die angegebenen Hilfsmittel – insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen – benutzt.

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Hamburg, den 02.12.2020