Topology Group
Institute for Algebra and Geometry
Summer term 2017

Arithmetic hyperbolic 3-manifolds

Research seminar

Introduction

The group $\mathrm{PSL}(2,\mathbb{C})$ can be identified with the group of orientation preserving isometries of hyperbolic 3-space. Accordingly, orientable, finite volume hyperbolic 3-manifolds correspond to discrete, torsion-free subgroups of $\mathrm{PSL}(2,\mathbb{C})$ with cofinite Haar measure. Mostow rigidity implies that these subgroups can be conjugated into $\mathrm{PSL}(2,k)$ where k is some *number field*: a finite field extension of $\mathbb{Q}$. This observation opens up deep and fruitful connections between hyperbolic geometry and algebraic number theory.

To name one, orders in quaternion algebras with a certain ramification behavior give rise to a sophisticated but constructive definition of *arithmetic hyperbolic 3-manifolds*. These manifolds constitute the most widely studied and best understood class of hyperbolic 3-manifolds. In this seminar we want to understand the definition, including the massive necessary background, and see some examples and applications. As a long term goal we want to see why this construction of arithmetic hyperbolic 3-manifolds coincides with the more familiar definition of arithmetic hyperbolic 3-manifolds coming from *arithmetic subgroups* of the *algebraic group* $\mathrm{PSL}(2,\mathbb{C})$.

References

Our main source is the Springer GTM "The Arithmetic of Hyperbolic 3-Manifolds" by Colin Maclachlan and Alan Reid [MR]. For background on algebraic number theory we recommend Jürgen Neukirch's treatise "Algebraische Zahlentheorie" [N] of which an English translation is available. An advanced book studying algebraic groups and arithmetic subgroups over general number fields is "Algebraic Groups and Number Theory" by Vladimir Platonov and Andrei Rapinchuk [PR].

Program

- *04/27.: Holger Kammeyer*
  Basics of algebraic number theory, part 1.
  Number fields, algebraic integers, Dedekind domains, units and class groups, ramification. [N, Chapter 1], [MR, Section 0.1-0.5]
- *05/04.: Holger Kammeyer*
  Basics of algebraic number theory, part 2.
  Valuations, completions, adèles and idèles, quadratic forms, the local-global principle. [N, Chapter 2], [MR, Section 0.6-0.9]

- *05/11.: Hartwig Senska*
  Kleinian groups and hyperbolic manifolds
  Elliptic, parabolic and loxodromic elements, reducible and elementary subgroups, cusps and geometrical finiteness, examples, volume. Mention hyperbolic Dehn surgery and rigidity as black box results. [MR, Chapter 1 with emphasis on 1.2, 1.3, 1.4.1 and 1.7]
- *05/18: Manuel Amann*
  Quaternion algebras I
  Quaternion algebras, norm and trace, split algebras and division algebras, orders, quaternion algebras as quadratic spaces, units mod center = orthognal group, classification over $\mathbb{R}$, $\mathfrak{p}$-adic fields and number fields. [MR, Sections 2.1-2.7]
- *06/01.: Caterina Campagnolo*
  Central simple algebras and the trace field
  CSAs and the Skolem Noether Theorem, trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$ is a number field and a homotopy invariant, quaternion algebra associated with Γ with coefficients in trace field, invariant trace field and invariant quaternion algebra, invariant trace fields of finite covolume Kleinian groups have a complex place, non-cocompact groups have matrix invariant quaternion algebras; quick report on generators. [MR, 2.8, 2.9 and Chapter 3]
- *06/08.: Sabine Braun*
  Examples
  Feel free to present your own selection of relevant examples from [MR, Chapter 4] but do cover Bianchi groups and the figure eight knot.
- *Wednesday, 06/14:* Benjamin Waßermann
  Applications
  Discreteness criteria, (non-)integral traces and geometric consequences, geodesics, surfaces and subgroup separability, geometry of the trace field. [MR, Chapter 5 (you may skip large parts in 5.2, put emphasis on 5.3, and only quickly report on 5.6)]
- *Friday, 06/30:* Federico Franceschini
  Orders in quaternion algebras
  Maximal orders and Eichler orders, localisation, local-global principle for maximality of orders, discriminants of orders, maximal orders over local fields: ramified case and split case, principal congruence subgroups, maximal order over global fields: discriminant criterion, type numbers of quaternion algebras and restricted class groups. [MR, Chapter 6]
- *07/06: Roman Sauer*
  Quaternion algebras II
  Groups of adèle and idèle points, behavior for field extensions, discreteness of k-points in adèle groups, duality for locally compact abelian groups, canonical characters, self-duality for local fields and adèle rings of quaternion algebras, existence of quaternion algebras with prescribed ramified places, splitting criterion; rough account on the Tamagawa number ($\operatorname{vol}(A^1_{\mathcal{A}}/A^1_k) = 1$ and this quotient is compact if A is a division algebra) and strong approximation. [MR Chapter 7]

- *07/13.: Michael Schrödl/Petra Schwer (?)*
  Arithmetic Kleinian groups
  Realifications of quaternion algebras over number fields, Main Theorem: norm one elements of orders in Eichler quaternion algebras give discrete finite covolume subgroups in products of SL_2*'s over local fields; definition of arithmetic Kleinian groups, necessity of required ramification structure, nonuniform arithmetic Kleinian groups = Bianchi groups, similar discussion for Fuchsian groups, identification theorem, invariant trace field and quaternion algebra form a complete commensurability invariant, the commensurator of arithmetic Kleinian groups. [MR Chapter 8, you may skip Section 8.5]*
- *07/20.: Michael Schrödl/Petra Schwer (?)*
  Arithmetic hyperbolic 3-manifolds and orbifolds
  A reassessment of the examples from the talk on 06/08 under the new arithmetic viewpoint. In particular, show that the figure eight knot group is the only arithmetic Kleinian group among all knot groups. Coordinate with the speaker on 06/08. [MR Chapter 9]
- *07/27: Roman Sauer*
  Discrete arithmetic groups
  Explain why the class of arithmetic Kleinian groups coincides with the class of arithmetic lattices in $\mathrm{PSL}(2,\mathbb{C})$ *in the sense of Margulis arithmeticity. [MR Chapter 10]*