Advanced Mathematics III
Exercise Sheet 4

**Keywords:** extrema, Lagrange multiplier, Banach’s fixed point theorem

**Exercise 1 (10 points)**
A wooden box without lid should be carved out of a wooden plate of area $A$ such that the volume of the box is maximal.

Calculate the edge lengths $x, y, z$ of the optimal box as a function of $A$.

**Proposal for a Solution to Exercise 1**

The function which should be maximised is $f(x, y, z) = xyz, x, y, z > 0$. The restriction is described by $h(x, y, z) = xy + 2(xz + yz) - A = 0$. Thus, the Lagrange function is given by $L = f + \lambda h$ with $\lambda \in \mathbb{R}$. A necessary condition for extremal points is $\nabla L = 0$ and $h = 0$.

Writing out those equations we have

\begin{align*}
(1) & \quad yz + \lambda(y + 2z) = 0 \\
(2) & \quad xz + \lambda(x + 2z) = 0 \\
(3) & \quad xy + 2\lambda(x + y) = 0 \\
(4) & \quad xy + 2(xz + yz) - A = 0
\end{align*}

We deduce

\[(2) - (1) : (z + \lambda)(x - y) = 0\]

Using (1) we can also deduce $z + \lambda \neq 0$ as otherwise we would have $z = 0$, which is a contradiction. Hence, $(2) - (1)$ implies $x = y$. Using (3) and $x > 0$ we derive $\lambda = -\frac{x}{4}$. Now with (2), we have $z = \frac{1}{2}x$. Finally, plugging everything into (4) yields $x = \sqrt[3]{\frac{A}{3}}, y = x$ and $z = \frac{1}{x}$. ♦
Exercise 2 (10 points)
A square box of volume 1l is to be built from cardboard. For stability reasons the top and the bottom of the box must contain five layers of cardboard, while the four sides are made from just one layer each. Determine the optimal lengths of the sides of the box such that the least amount of cardboard is to be used.

Proposal for a Solution to Exercise 2
We denote by \(x, y, z > 0\) the edge length (in decimeter) of the box. The function to minimise is
\[
f(x, y, z) = 10xy + 2yz + 2xz
\]
(two five layer plus four single layer). The restriction is that the box must have volume \(1l\) and is given by the function
\[
h(x, y, z) = xyz - 1 = 0
\]
Using the Lagrangian \(L(x, y, z, \lambda) = f(x, y, z) + \lambda h(x, y, z)\), we derive the necessary condition \(L' = 0\), i.e.
\[
\begin{align*}
(1) & \quad L_x = 10y + 2z + \lambda yz = 0 \\
(2) & \quad L_y = 10x + 2z + \lambda xz = 0 \\
(3) & \quad L_z = 2y + 2x + \lambda xy = 0 \\
(4) & \quad L_\lambda = xyz - 1 = 0.
\end{align*}
\]
The difference (2) \( - (1)\) yields \((10 + \lambda z)(x - y) = 0\). Since \(\lambda z = -10\) in (1) yields \(z = 0\), which is a contradiction, we must have \(x = y\). From (3), \(x = y\) and \(x > 0\) we derive \(\lambda = -\frac{4}{x}\). Plugging those in into 2 yields \(x = \frac{1}{5}z\) and with (4) follows:
\[
x = \sqrt[3]{\frac{1}{5}} \text{ dm}, \quad y = x, \quad z = 5x
\]
for the optimal edge lengths.
\[
\diamondsuit
\]
Exercise 3 (10 points)
Let a scalar-valued function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and a set \( M \subset \mathbb{R}^2 \) be given by
\[
f(x, y) = x^2 y \quad \text{and} \quad M = \{(x, y) \in \mathbb{R}^2 : x^2 + y \leq 3\}.
\]
a) Give reasons for why for \( f \) there exists a global maximum. (Attention: \( M \) here is unbounded!)
b) Does \( f \) also possess a global minimum? Give reasons.
c) Determine all local and global extremal positions and extremal values of \( f \), and provide arguments for their existence. (Hint: Begin by looking for extrema of \( f \) in the interior of \( M \) first.)

Proposal for a Solution to Exercise 3

a) As \( x^2 \geq 0 \), the function \( f(x, y) = x^2 y \) is monotonically increasing with respect to \( y \) for a fixed \( x \in \mathbb{R} \). Hence, we can restrict our search for a global maxima of \( f \) to the set
\[
N := M \cap \{(x, y) \in \mathbb{R}^2 : y \geq 0\} = \{(x, y)^\top \in \mathbb{R}^2 : x^2 + y \leq 3, \ y \geq 0\}
\]
\( N \) contains the point \((0,0)^\top\) and is thus non-empty. Let \((\hat{x}, \hat{y})\) be a global maxima of \( f \) in \( N \). Then it is also a global maxima of \( f \) in \( M \) since for \((x, y)^\top \in M \setminus N\) we have
\[
f(x, y) = x^2 y \leq x^2 \cdot 0 = 0 = f(0,0) \leq f(\hat{x}, \hat{y})
\]
\( N \) is a bounded set because for \((x, y)^\top \in N\) we have
\[
x^2 \leq x^2 + y \leq 3, \quad \text{thus} \quad |x| \leq \sqrt{3}
\]
and
\[
0 \leq y \leq x^2 + y \leq 3.
\]
Because \( N \) is the intersection of two closed sets, it is also closed. Hence, \( N \) is compact. Since \( f \) is continuous, there exists a global maximum of \( f \) on \( N \), which is by above reasoning also a global maximum of \( M \).

b) The function \( f \) does not have a global minimum on \( M \). Indeed, the points \((1, -n)^\top\) with \( n \in \mathbb{N} \) are all contained in \( M \) as
\[
1^2 + (-n) = 1 - n \leq 1 - 1 = 0 \leq 3, \quad n \in \mathbb{N},
\]
and their values
\[
f(1, -n) = -n \quad \text{with} \ n \in \mathbb{N}
\]
are unbounded from below.

c) The interior of \( M \) is
\[
\overset{\circ}{M} = \{(x, y)^\top \in \mathbb{R}^2 : x^2 + y < 3\}.
\]
Every extremal point \((x, y)^\top\) of \( f \) in the interior of \( M \) satisfies
\[
\nabla f(x, y) = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
The first equation is equivalent to \( x = 0 \) or \( y = 0 \) and the second one to \( x = 0 \). Since both conditions have to be fulfilled, we have – using the definition of \( \hat{M} \) – that the set of critical points of \( f \) is given by

\[
K := \{(0, y) \in \mathbb{R}^2 : y < 3\}
\]

We are now investigating which of these points are extremal points. For this let \((0, \hat{y})\) be a point of \( K \) (the one which we are examining).

**Case** \( \hat{y} < 0 \): If \((x, y) \in M\) is close to \((0, \hat{y})\), then we have \( y < 0 \). Since \( x^2 \) is non-negative, we have

\[
f(x, y) = x^2 y \leq x^2 \cdot 0 = 0 = f(0, \hat{y}).
\]

Hence, \((0, \hat{y})\) is a local maxima with value 0.

**Case** \( \hat{y} > 0 \): Similar to the previous case, if \((x, y) \in M\) is close to \((0, \hat{y})\), then we have \( y > 0 \). Since \( x^2 \) is non-negative, we have

\[
f(x, y) = x^2 y \geq x^2 \cdot 0 = 0 = f(0, \hat{y}).
\]

Hence, \((0, \hat{y})\) is a local minima with value 0.

**Case** \( \hat{y} = 0 \): We conjecture that \((0, 0)^\top\) is a saddle point. To prove this, we define the following two sequences

\[
a_n := \left(\frac{1}{n}, -\frac{1}{n}\right)^\top, \quad b_n := \left(\frac{1}{n}, \frac{1}{n}\right)^\top,
\]

for \( n \in \mathbb{N} \). Both sequences converge to \((0, 0)^\top\) for \( n \) to infinity. Therefore, every neighbourhood of \((0, 0)^\top\) contains infinitely many sequence elements of \( a_n \) and \( b_n \). We have \( f(a_n) = -\frac{1}{n^3} < 0 \) and \( f(b_n) = \frac{1}{n^3} > 0 \) for all \( n \in \mathbb{N} \). Thus, \((0, 0)^\top\) can neither be a maximum nor a minimum and is hence a saddle point.

We continue our search for extrema of \( f \) on the boundary of \( M \) which is

\[
\partial M = \{(x, y) \in \mathbb{R}^2 : x^2 + y = 3\}
\]

Every extremal point of \( f \) on \( M \) which lies in \( \partial M \), is in particular an extremal point of \( f \) under the condition

\[
g(x, y) := x^2 + y - 3 = 0.
\]

The method of choice is the Lagrange multiplier, which we can apply since

\[
\nabla g(x, y) = \begin{pmatrix} 2x \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}
\]

and \( \nabla g(x, y) \) is linearly independent.

With the remarks above, the Lagrangian is

\[
L(x, y, \lambda) = x^2 y + \lambda(x^2 + y - 3).
\]

By the Lagrange multiplier theorem, for an extremal point \((x, y)^\top\) there exists a Lagrange multiplier \( \lambda \in \mathbb{R} \) with

\[
\nabla L(x, y, \lambda) = \begin{pmatrix} 2x(y + \lambda) \\ x^2 + \lambda \\ x^2 + \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
Case \( x = 0 \): By the third equation we have \( y = 3 \) and the critical point is \((0,3)^\top\).

Case \( x \neq 0 \): From the first equation it follows \( \lambda = -y \), further from the second one \( x^2 = y \), and finally from the third one \( x = \pm \sqrt{\frac{3}{2}} \) with \( y = \frac{3}{2} \) in both cases.

Therefore, the critical points are \((\sqrt{\frac{3}{2}}, \frac{3}{2})\) and \((-\sqrt{\frac{3}{2}}, \frac{3}{2})\).

We have

\[
f(0,3) = 0, \quad f\left(-\sqrt{\frac{3}{2}}, \frac{3}{2}\right) = \frac{9}{4}, \quad f\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right) = \frac{9}{4}.
\]

By part a), \( f \) has on \( M \) a global maximum. Since all critical points in the interior of \( M \) have function value 0, which is smaller then \( \frac{9}{4} \), the points \((-\sqrt{3/2}, 3/2)^\top\) and \((\sqrt{3/2}, 3/2)^\top\) must be the global maxima and in particular also local maxima. Finally, the point \((0,3)^\top\) is a local minima as can be seen by the same argument as given above in the case \( \hat{y} > 0 \).

\[\blacksquare\]
Exercise 4 (10 points)

An ostrich egg is placed on a cooker. The surface of the egg can be described by the function
\[ \frac{1}{6}x^2 + y^2 + z^2 = 1 \] (in units of 10 cm)

After a few minutes, the temperature distribution inside the egg is given by
\[ f(x, y, z) = 60 - 5 \left( x^2 + y^2 + (z + 1)^2 \right) \] (in degree Celsius)

Determine the place of the egg where the temperature is minimal. What temperature does that point have?

Proposal for a Solution to Exercise 4

The egg is closed and bounded and since \( f \) is continuous, there exists a global minimum (and maximum) of \( f \).

Can the minimum be in the interior of the egg? Critical points of \( f \) satisfy
\[
\nabla f(x, y, z) = \begin{pmatrix}
-10x \\
-10y \\
-10(z + 1)
\end{pmatrix} = 0 \iff x = 0, y = 0, z = -1
\]

Therefore, a minimum is only possible in the point \((0, 0, -1)\), however, this point does not lie in the interior of the egg but on the boundary. Hence, there is no minimum in the interior of the egg.

If we look for minima on the boundary of the egg, we minimise
\[ f(x, y, z) = 60 - 5(x^2 + y^2 + (z + 1)^2) \]

under the condition
\[ g(x, y, z) := \frac{1}{6}x^2 + y^2 + z^2 - 1 = 0. \]

First we note that \( \nabla g(x, y, z) = (\frac{1}{3}x, 2y, 2z)^T \) becomes zero if and only if \( x = y = z = 0 \). However, \( x = y = z = 0 \) does not satisfy the condition \( g(x, y, z) = 0 \), i.e. does not lie on the boundary of the egg. Therefore, the set \( \{\nabla g(x, y, z)\} \) is always linearly independent on the boundary of the egg.

Hence, we can determine critical points using Lagrange multipliers. The Lagrangian is
\[ L(x, y, z, \lambda) := f(x, y, z) - \lambda \cdot g(x, y, z). \]

Critical points satisfy
\[
\nabla h(x, y, z, \lambda) = \begin{pmatrix}
-10x - \frac{1}{3} \lambda x \\
-10y - 2\lambda y \\
-10(z + 1) - 2\lambda z
\end{pmatrix} = \begin{pmatrix}
-x(10 + \frac{1}{3} \lambda) \\
-y(10 + 2\lambda) \\
-z(10 + 2\lambda) - 10
\end{pmatrix} = 0.
\]

We have
\[
-z(10 + 2\lambda) = 10 \Rightarrow 10 + 2\lambda \neq 0 \\
-y(10 + 2\lambda) = 0 \Rightarrow y = 0 \\
x(10 + \frac{1}{3} \lambda) = 0 \Rightarrow x = 0 \text{ oder } \lambda = -\frac{30}{7}
\]
Case $x = 0$: Plugging the values in into $g$ yields
\[ g(0, 0, z) = z^2 - 1 = 0 \implies z = \pm 1. \]

The values of $f$ at those points are $f(0, 0, 1) = 40$ and $f(0, 0, -1) = 60$.

Case $\lambda = -30$: We have $\lambda = -30 \implies z = \frac{-10}{10 + 60} = \frac{-10}{70} = \frac{1}{7}$. Plugging the values in into $g$ yields
\[ g(x, 0, \frac{1}{7}) = \frac{1}{6}x^2 + \left(\frac{1}{7}\right)^2 - 1 = 0 \iff \frac{1}{6}x^2 = 1 - \frac{1}{25} = \frac{24}{25} = \frac{64/25}{25} \iff x = \pm \sqrt{\frac{64/25}{25}} = \pm \frac{12}{5}. \]

The value of $f$ at those points are
\[ f(\pm \frac{12}{5}, 0, \frac{1}{7}) = 60 - 5\left(\frac{144}{25} + \frac{36}{25}\right) = 60 - \frac{180}{5} = 60 - 36 = 24. \]

We conclude that the coldest points of the egg are $(\pm \frac{12}{5}, 0, \frac{1}{7})$ with a temperature of $24^\circ C$. ♦
Exercise 5  (10 points)

We want to calculate the intersection point of the line $y = x + 1$ and the graph of the function $f(x) = \tan(x)$. The calculation should be done using Banach’s fixed point theorem.

a) Why does the fixed point iteration with start point $x_0 = 1$ and fixed point equation $x = \tan(x) - 1$ not converge? State the first five sequence elements of the above iteration.

b) Consider the fixed point equation $\arctan(x + 1) = x$ and show that the fixed point iteration converges in this case. How many iterations are sufficient to calculate the fixed point $\hat{x}$ up to the second decimal place?

Proposal for a Solution to Exercise 5

a) The fixed point equation $x = \tan x - 1$ yields the iteration formula

$$x_{n+1} = g(x_n) := \tan x_n - 1.$$  

However, we have $g'(x) = 1 + \tan^2(x) > 1$. Thus, $g$ is not a contraction and the iteration diverges. For the first few steps we have the value

$x_0 = 1, x_1 = 0.5574, x_2 = -0.3767, x_3 = -1.3955, x_4 = -6.647$.

b) For this fixed point equation, the iteration formula is

$$x_{n+1} = h(x_n) := \arctan(x_n + 1)$$

The derivative of $h$ is $h'(x) = \frac{1}{1+(x+1)^2}$.

The application of Banach’s fixed point theorem requires a contraction constant $\kappa$, which is strictly small than 1.

We choose $I = [1, 2]$ as our domain and check the requirements of Banach’s fixed point theorem:

contraction: For $x \in I$ we have $h'(x) \leq \frac{1}{2}$, i.e. $h$ is a contraction on $I$ with contraction constant $\kappa = \frac{1}{2} < 1$.

maps $I$ onto itself: We need to show that $h(x) \in I$, i.e. $1 \leq h(x) \leq 2$, for all $x \in I = [1, 2]$. From $h'(x) > 0$ we deduce that $h$ is monotonically increasing. Thus, $1 < 1.1071 = h(1) \leq h(x) \leq h(2) = 1.2490 < 2$.

The convergence of the iteration follows now from Banach’s fixed point theorem. For the starting value $x_0 = 1$ we get $x_1 = 1.1071$ after the first step. The error can now be bound by the a-priori estimate:

$$|x_k - \hat{x}| \leq \frac{1}{2^k(1 - \frac{1}{2})}|1.1071 - 1| < 0.3 \frac{1}{2^k}$$

If we require $\frac{0.3}{2^k} < 10^{-2}$, then we have

$$k > \log(10^{-2}/0.3)/\log(1/2) \approx 4.9,$$
Thus, $x_5$ is sufficiently precise to calculate $\hat{x}$ up to the second decimal place. Indeed, the first few iteration steps yield $x_2 = 1.1277$, $x_3 = 1.1314$, $x_4 = 1.1321$, $x_5 = 1.1322$. Hence, the intersection point is approximately $(1.13, 2.13)$. We can also see that the a-priori estimate is quite rough. In this case $x_3$ would have already been sufficiently precise for the second decimal place.