Advanced Mathematics III
Exercise Sheet 5

Keywords: Banach’s fixed point theorem, Newton’s method, iterated integrals, Fubini

Exercise 1 (10 points)
Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by
$$f(x_1, x_2) = \left( \frac{(1 - x_2^2)^{1/2}}{\sqrt{21}(9 - 5x_1^2)^{1/2}} \right).$$
Show that $f$ has a fixed point on $M = [4/5, 1] \times [0, 3/5] \subset \mathbb{R}^2$ using Banach’s fixed point theorem.

Exercise 2 (10 points)
Let a vector-valued function $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$f(x, y) = \left( x^2 + 10y - 5 \right).$$

a) Determine the general iteration procedure of the Newton method for solving the problem $f(x, y) = (0, 0)^T$ for the function $f$.

b) Perform four iterations of the Newton method using the initial value $(x(0), y(0)) = (4, -3)$. Use a pocket calculator to compute the solution to an accuracy of 5 decimal places.

Exercise 3 (10 points)
Calculate the integral
$$\iint_B (x^2 - y^2) \, d(x, y)$$
over the area $B$, which is bounded by the two curves $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$, by first integrating

(a) with respect to $x$,

(b) with respect to $y$. 

**Exercise 4** (10 points)
We denote by $D$ the triangular area with corners $(0, 0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $(\pi, 0)$. Calculate the domain integral

$$J = \int_D \sqrt{\sin x_1 \sin x_2 \cos x_2} \, d(x_1, x_2)$$

**Exercise 5** (10 points)

a) Calculate the following integrals. What is the relation to Fubini’s theorem?

$$\int_0^1 \int_1^2 \frac{s}{s^2 + t} \, dt \, ds , \quad \int_1^2 \int_0^1 \frac{s}{s^2 + t} \, ds \, dt .$$

b) Evaluate the following iterated integral:

$$\int_0^1 \int_0^4 \exp(s \sqrt{t}) \, dt \, ds$$

**Hand in** your solutions in the exercise class or the lecture on Tuesday, 3.12.2019.