Advanced Mathematics III
Exercise Sheet 6

Keywords: polar/spherical/cylindrical coordinates, transformation formula

Exercise 1 (10 points)
Let $B$ be a spherical shell centred on the origin, with outer radius $R$ and inner radius $r$ ($r < R$). Evaluate the domain integral

$$\int_B \sqrt{x_1^2 + x_2^2 + x_3^2} \, d(x_1, x_2, x_3).$$

Exercise 2 (10 points)
The semi-ball $B = \{x \in \mathbb{R}^3 : |x| \leq R, z \geq 0\}$ is equipped with the density $\rho(x) = a x_3$ where $a > 0$ is a constant. Calculate the mass and the centre of gravity of the semi-ball.

Exercise 3 (10 points)
The two curves in $\mathbb{R}^2$ parametrised by $x(t) = (\sin t, \cos t)^T$ and $y(t) = (1 + \sin t)(\sin t, \cos t)^T$ with $t \in [0, \pi/2]$ bound a domain $B$ in the first quadrant ($x_1, x_2 > 0$) of the plane. Calculate the integral

$$\int_B \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} \, dx.$$

using a suitable coordinate transformation.

Exercise 4 (10 points)
We cut a solid out of a (infinite) cylinder in $\mathbb{R}^3$. The cylinder is given by $x_1^2 + x_2^2 \leq 9$, the first cut is given by the $x_1$-$x_2$-plane, and the second one by the surface given by $x_3 = e^{x_1^2 + x_2^2}$. The solid itself has a density given by $\rho(x) = x_2^2$. What mass does the solid have and what is its centre of gravity?
Exercise 5 (10 points)

Determine the coordinates of the centre of gravity,

\[ x_S = \frac{1}{G} \iiint_B x \, d(x, y), \quad y_S = \frac{1}{G} \iiint_B y \, d(x, y), \quad G = \iiint_B d(x, y) \]

of a lemniscate's wing

\[ B : (x^2 + y^2)^2 \leq 2(x^2 - y^2), \quad x \geq 0, \]

employing the coordinate transformation

\[ \Phi : x(u, v) = u \sqrt{2} \cos 2v \cos v, \quad y(u, v) = u \sqrt{2} \cos 2v \sin v, \quad 0 \leq u \leq 1, \quad -\frac{\pi}{4} \leq v \leq \frac{\pi}{4}. \]

Hand in your solutions in the exercise class or the lecture on Tuesday, 10.12.2019.