Advanced Mathematics III  
Exercise Sheet 9

Keywords: curl, Stokes’ theorem, Green’s identities

Exercise 1 (10 points)
Consider the velocity field \( u(x, y, z) = (z - y, x - z, y - x)^\top \) of a turbulent flow, and a surface given by

\[
F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x^2 + y^2 \leq 4, z = xy \right\}.
\]

a) Sketch the surface \( F \).

b) Evaluate on \( F \) the integral over all vortex strengths of \( u \):
\[
\int_F \text{rot} u \cdot d\sigma.
\]

c) Compute the circulation of \( u \), \( \int_{\partial F} u \cdot ds \), along the boundary curve \( \partial F \), and verify thus Stokes’ integral theorem in \( \mathbb{R}^3 \).

Proposal for a Solution to Exercise 1

a) 

b) We can parametrise the surface \( F \) as follows:
\[
\Phi(x, y) = (x, y, xy)^\top \quad \text{with} \quad (x, y) \in K := \left\{ (x, y)^\top \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \right\}.
\]

We have
\[
\left( \Phi_x \times \Phi_y \right)(x, y) = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} = \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix}.
\]

Thus, the normal vector is pointing upwards in positive \( z \)-direction.

The curl of \( u \) is
\[
\text{rot} u(x, y, z) = (1 - (-1), 1 - (-1), 1 - (-1))^\top = (2, 2, 2)^\top.
\]
Hence,

\[
\iint_F \text{rot} \, u \cdot d\sigma = \iint_K \begin{pmatrix} 2 & -y \\ 2 & -x \\ 1 & \end{pmatrix} \, d(x, y) = 2 \iint_K 1 - x - y \, d(x, y)
\]

\[
= \text{polarkoord. } 2 \int_0^{2\pi} \int_0^2 (1 - r \cos \varphi - r \sin \varphi) \, r \, dr \, d\varphi = 2 \int_0^2 \pi r \, dr = 8\pi.
\]

c) The parametrisation of the boundary curve \( \partial F \) must be positively oriented, i.e. the surface \( F \) must be on the left side (w.r.t. the normal vector of the surface).

\[
c(t) = (2 \cos t, 2 \sin t, 4 \cos t \sin t)^T \quad \text{for} \quad t \in [0, 2\pi].
\]

We have \( \dot{c}(t) = (-2 \sin t, 2 \cos t, 4 \left( \cos^2 t - \sin^2 t \right))^T \). Thus, the circulation is

\[
\int_{\partial F} u \cdot ds = 4 \int_0^{2\pi} \begin{pmatrix} 2 \cos t \sin t - \sin t \\ \cos t - 2 \cos t \sin t \\ \sin t - \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 2 \left( \cos^2 t - \sin^2 t \right) \end{pmatrix} \, dt
\]

\[
= 4 \int_0^{2\pi} 1 - 2 \sin^3 t - 2 \cos^3 t \, dt = 4 \int_0^{2\pi} 1 \, dt = 8\pi.
\]

\[\text{♦}\]
Exercise 2 (10 points)
Consider a cylinder \( Z = \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 4, |x_3| \leq 2 \} \) and a surface defined by

\[
S := \partial Z \setminus \{ x \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq 1, x_2 > 0 \}.
\]

Evaluate

\[
\int_S \text{rot} F(x) \cdot \nu_S(x) \, d\sigma
\]

for a differentiable vector field \( F(x) = (x_1, x_2, x_3, -x_1^2 x_3)^T, x \in \mathbb{R}^3 \) and an outward-oriented unit normal vector field \( \nu_S \) by means of Stokes' integral theorem.

Proposal for a Solution to Exercise 2

Stokes' theorem states

\[
\oint_S \text{rot} F(x) \cdot \nu_S(x) \, d\sigma = \int_{\partial S} F(x) \cdot d\ell,
\]

i.e. the surface integral can be calculated by means of the line integral over the boundary of \( S \).

The surface \( S \) looks like a cylindrical closed tin can with radius 2 and height 4. Half way up on its surface is a hole punched out by a cylindrical tool of radius 1. The boundary of this hole is the boundary \( \partial S \) of the surface \( S \). As a set we can describe this boundary as

\[
\partial S = \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 = 4, x_1^2 + x_3^2 = 1, x_2 > 0 \}.
\]

To find a parametrisation of this boundary, we are using cylinder coordinates w.r.t. the \( x_2 \)-axis (because the boundary winds around the \( x_2 \)-axis):

\[
x_1 = r \sin \varphi, \quad x_2 = z, \quad x_3 = r \cos \varphi, \quad 0 \leq r, \ 0 \leq \varphi \leq 2\pi, \ z \in \mathbb{R}
\]

Using these coordinates, we can rewrite the above conditions of \( \partial S \) to

\[
\begin{align*}
x_1^2 + x_2^2 &= 4, \quad x_1^2 + x_3^2 = 1, \quad x_2 > 0 \\
\iff (r \sin \varphi)^2 + z^2 &= 4, \quad (r \sin \varphi)^2 + (r \cos \varphi)^2 = 1, \quad z > 0 \\
\iff z^2 &= 4 - r^2 \sin^2 \varphi, \quad r = 1, \quad z > 0 \\
\iff z &= \sqrt{4 - \sin^2 \varphi}, \quad r = 1.
\end{align*}
\]

Hence,

\[
x(\varphi) := \begin{pmatrix} \sin \varphi \\ \sqrt{4 - \sin^2 \varphi} \\ \cos \varphi \end{pmatrix}, \quad \varphi \in [0, 2\pi]
\]

is a parametrisation of \( \partial S \).

We have to check whether this is a positively oriented parametrisation. By assumption the surface \( S \) is oriented such that the normal vector points outwards, i.e. away from the origin. Using the above parametrisation, we see that the surface lies right of the
boundary. Hence, the above parametrisation is negatively oriented. To apply Stokes’ theorem, we need a positively oriented parametrisation. We get such a parametrisation by replacing \( \varphi \) with \(-\varphi\). Alternatively, we can use the above negatively oriented parametrisation and negate the line integral. The latter is what we are going to do:

\[
\iint_S \text{rot} \mathbf{F}(x) \cdot \mathbf{n}_S(x) \, d\sigma = \oint_{\partial S} \mathbf{F}(x) \cdot d\ell = -\int_0^{2\pi} \mathbf{F}(x(\varphi)) \cdot \dot{x}(\varphi) \, d\varphi
\]

\[
= -\int_0^{2\pi} \left( \begin{array}{c} \sin \varphi \cos^2 \varphi \\ \sin \varphi \cos \varphi \sqrt{4 - \sin^2 \varphi} \\ -\sin^2 \varphi \cos \varphi \end{array} \right) \cdot \left( \begin{array}{c} \cos \varphi \\ \frac{\sin \varphi \cos \varphi}{\sqrt{4 - \sin^2 \varphi}} \\ -\sin \varphi \end{array} \right) \, d\varphi
\]

\[
= -\int_0^{2\pi} \sin \varphi \cos \varphi \cos \varphi - \sin^2 \varphi \cos \varphi + \sin^3 \varphi \cos \varphi \, d\varphi
\]

\[
= -\int_0^{2\pi} \sin \varphi \cos \varphi \cos \varphi - \sin^2 \varphi \cos \varphi \, d\varphi
\]

\[
= -\left[ \int_0^{\frac{\pi}{4}} \sin^2 t \, dt \right]_{\frac{\pi}{4}}^{\frac{2\pi}{4}} + \int_0^{2\pi} \frac{1}{4} (1 - \cos(2\varphi))(1 + \cos(2\varphi)) \, d\varphi
\]

\[
= \frac{1}{4} \int_0^{2\pi} 1 - \cos^2(2\varphi) \, d\varphi
\]

\[
= \frac{1}{8} \int_0^{2\pi} 1 - \cos(4\varphi) \, d\varphi
\]

\[
= \frac{1}{8} \left[ \varphi - \frac{1}{4} \sin(4\varphi) \right]_0^{2\pi}
\]

\[
= \frac{\pi}{4}
\]

\[\square\]
Exercise 3 (10 points)

Let scalar–valued functions \( f(x) = x_1^2 + x_2^2 \) and \( g(x) = (x_1^2 + x_2^2 - 4)^2 \) be given, as well as the disc \( K \subseteq \mathbb{R}^2 \) of radius 2 centred at the origin. Evaluate the domain integral

\[
\frac{1}{16} \iint_K g(x) \Delta f(x) \, dx.
\]

Hint: Make use of Green’s second integral formula.

Proposal for a Solution to Exercise 3

By the second identity of Green we have

\[
\iint_K g(x) \Delta f(x) \, dx = \int_{\partial K} [g(x) (\nabla f(x) \cdot \nu(x)) - f(x) (\nabla g(x) \cdot \nu(x))] \, ds + \iint_K f(x) \Delta g(x) \, dx.
\]

On the boundary \( \partial K \) we have \( x_1^2 + x_2^2 = 4 \). Hence, \( g \) vanishes on \( \partial K \) and we have

\[
\int_{\partial K} g(x) (\nabla f(x) \cdot \nu(x)) \, ds = 0.
\]  \hspace{1cm} (1)

Furthermore, we have \( \nabla g(x) = (x_1^2 + x_2^2 - 4) (4x_1, 4x_2)^T \). Therefore, also \( \nabla g(x) \) vanishes on \( \partial K \) and we have

\[
\int_{\partial K} f(x) (\nabla g(x) \cdot \nu(x)) \, ds = 0.
\]

All that is left is the last integral over \( K \) with \( f \) and \( \Delta g(x) = 16(x_1^2 + x_2^2 - 2) \). We calculate it using polar coordinates:

\[
\frac{1}{16} \iint_K g(x) \Delta f(x) \, dx = \frac{1}{16} \iint_K f(x) \Delta g(x) \, dx
\]

\[
= \frac{1}{16} \iint_K x_1^2 (x_1^2 + x_2^2) (x_1^2 + x_2^2 - 2) \, dx
\]

\[
= \frac{1}{16} \int_{r=0}^{2} \int_{\varphi=0}^{2\pi} r^2 (\cos^2 \varphi) (r^2 - 2) r \, d\varphi \, dr
\]

\[
= \frac{1}{16} \int_{\varphi=0}^{2\pi} \cos^2 \varphi \, d\varphi \int_{r=0}^{2} (r^6 - 2r^4) \, dr
\]

\[
= \frac{1}{16} \left[ \frac{1}{2} (\varphi + \cos \varphi \sin \varphi) \right]_{\varphi=0}^{2\pi} \left[ \frac{1}{7} r^7 - \frac{2}{5} r^5 \right]_{r=0}^{2} = \frac{12}{35} \pi.
\]

\♦
Exercise 4 (10 points)
Consider a domain \( D = \{ x \in \mathbb{R}^2 : x_2 > 0, \ 1 < x_1^2 + x_2^2 < 4 \} \) and scalar-valued functions \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x_1, x_2) = x_1^2 x_2 - \frac{x_2^2}{2} \) and \( g : \mathbb{R}^2 \to \mathbb{R}, \ g(x_1, x_2) = x_1 + 1 \). Use Green’s first identity to evaluate the line integral \( \int_{\partial D} g(x) \frac{\partial f}{\partial \nu}(x) \, d\ell \), where \( \nu \) denotes the outward-pointing unit normal vector to \( \partial D \).

Proposal for a Solution to Exercise 4
By Green’s first identity we have
\[
\int_{\partial D} g(x) \frac{\partial f}{\partial \nu}(x) \, d\ell = \iint_D \left( g(x) \Delta f(x) + \nabla g(x) \cdot \nabla f(x) \right) \, dx.
\]
We also have
\[
\nabla g(x) = (1, 0)^T, \\
\nabla f(x) = (2x_1x_2, x_1^2 - x_2)^T, \\
\Delta f(x) = \nabla \cdot \nabla f(x) = 2x_2 - 1.
\]
Hence,
\[
\int_{\partial D} g(x) \frac{\partial f}{\partial \nu}(x) \, d\ell = \iint_D (4x_1x_2 + 2x_2 - x_1 - 1) \, dx.
\]
We use polar coordinates to calculate the domain integral on the right (the functional determinant is \( r \)). With \( B = (1, 2) \times (0, \pi) \) and \( x_1 = r \cos(\varphi), \ x_2 = r \sin(\varphi) \) we have
\[
\iint_D (4x_1x_2 + 2x_2 - x_1 - 1) \, dx = \iint_B \left( 4r^3 \cos(\varphi) \sin(\varphi) \cos(\varphi) + 2r^2 \cos(\varphi) - 2r \cos(\varphi) - r \right) \, d(r, \varphi)
\]
\[
= \int_0^2 \int_0^\pi \left( 4r^3 \cos(\varphi) \sin(\varphi) \cos(\varphi) + 2r^2 \cos(\varphi) - 2r^2 \cos(\varphi) - r \right) \, d\varphi \, dr
\]
\[
= \int_0^2 \left[ 2r^3 \sin^2(\varphi) - 2r^2 \cos(\varphi) - 2r^2 \sin(\varphi) - r\varphi \right]_0^\pi \, dr
\]
\[
= \int_0^2 \left( 4r^2 - \pi r \right) \, dr = \left[ \frac{4}{3} r^3 - \frac{\pi}{2} r^2 \right]_1^2
\]
\[
= \frac{32}{3} - \frac{4}{3} \pi(4 - 1) = \frac{28}{3} - \frac{3\pi}{2}.
\]
We are allowed to apply Fubini’s theorem since the integrand is continuously extendable to the compact set $\bar{B}$, in particular integrable. By Fubini’s theorem the order of the integration does not matter, when choosing a different order we get

$$\iint_D (4x_1x_2 + 2x_2 - x_1 - 1) \, dx = \int_0^\pi \int_1^{\sqrt{2}} \left( 4r^3 \sin(\varphi) \cos(\varphi) + 2r^2 \sin(\varphi) - r^2 \cos(\varphi) - r \right) dr \, d\varphi$$

$$= \int_0^\pi \left[ r^4 \sin(\varphi) \cos(\varphi) + \frac{2}{3} r^3 \sin(\varphi) - \frac{1}{3} r^3 \cos(\varphi) - \frac{1}{2} r^2 \right]_1^2 \, d\varphi$$

$$= \int_0^\pi \left( 15 \sin(\varphi) \cos(\varphi) + \frac{14}{3} \sin(\varphi) - \frac{7}{3} \cos(\varphi) - \frac{3}{2} \right) \, d\varphi$$

$$= \left[ \frac{15}{2} \sin^2(\varphi) - \frac{14}{3} \cos(\varphi) - \frac{7}{3} \sin(\varphi) - \frac{3}{2} \varphi \right]_0^\pi = \frac{28}{3} - \frac{3\pi}{2}.$$
Exercise 5 (10 points)

Let a Möbius strip \( M \) be given by the parametrisation
\[
X(u, v) = \left( (2 + u \cos v) \cos(2v), (2 + u \cos v) \sin(2v), u \sin v \right)^\top, \quad 0 \leq u \leq 1, 0 \leq v \leq 2\pi,
\]
and a vector field \( V \) by
\[
V(x) = \left( -\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right)^\top, \quad (x_1, x_2) \neq (0, 0).
\]

Evaluate the two integrals
\[
\iint_M \text{rot} \, V(x) \cdot \nu(x) \, d\sigma \quad \text{and} \quad \int_{\partial M} V(x) \cdot ds,
\]
where \( \nu \) denotes the normal vector to \( M \). Why can you not apply Stokes’ integral theorem in this context?

Proposal for a Solution to Exercise 5

First we notice that \( \text{rot} \, V = 0 \). Hence, the first integral is zero.

For the second integral we need a parametrisation of the boundary \( \partial M \). Considering the parametrisation of \( M \) we have two choices: \( u = 0 \) or \( u = 1 \). The choice \( u = 0 \) yields a circle of radius 2, which is traversed twice. This is not the boundary of \( M \). The choice \( u = 1 \), however, yields a parametrisation of the boundary \( \partial M \).

\[
X(t) = \left( \frac{(2 + \cos t) \cos(2t)}{\sin t}, \frac{(2 + \cos t) \sin(2t)}{\sin t} \right), \quad 0 \leq t \leq 2\pi.
\]

Its derivative is
\[
\dot{X}(t) = \left( -\frac{\sin t \cos(2t)}{\cos(t)} - 2(2 + \cos(t)) \sin(2t) \right), \quad -\sin t \sin(2t) + 2(2 + \cos(t)) \cos(2t) \right). \quad -\sin t \sin(2t) + 2(2 + \cos(t)) \cos(2t) \right), \quad -\sin t \sin(2t) + 2(2 + \cos(t)) \cos(2t) \right).
\]

Thus, we have for the line integral
\[
\int_{\partial M} V(x) \cdot ds = \int_0^{2\pi} \left( -\frac{\sin(2t)}{2 + \cos(t)} - \frac{\cos(2t)}{2 + \cos(t)} \right) dt
= \int_0^{2\pi} \left( \frac{\sin(2t) \sin t \cos(2t)}{2 + \cos(t)} + 2 \sin(2t)^2 - \frac{\cos(2t) \sin t \sin(2t)}{2 + \cos(t)} + 2 \cos(2t)^2 \right) dt
= 2 \int_0^{2\pi} 1 \, dt = 4\pi.
\]

In particular the two integrals do not coincide!
Why does this not contradict Stokes’ theorem? To apply Stokes’ theorem we need a smooth surface.

Recall the definition of a smooth surface: \( B \) an open set, \( X : B \rightarrow \mathbb{R}^3 \) continuously differentiable such that the partial derivatives are linearly independent for every point in \( B \). Furthermore, the unit normal vector \( \nu = \frac{1}{||X_u \times X_v||} X_u \times X_v \) must be continuously extendable to \( \overline{B} \).

In our case we have \( B = (0,1) \times (0,2\pi) \). The Möbius strip is in the sense of the above definition not a smooth surface because the unit normal vector cannot be extended continuously to the boundary of \( B \). Note that parts of the boundary of \( B \) belong to the interior of the Möbius strip, namely \( u = 0 \) and \( v = 0, 2\pi \). Only \( u = 1 \) results in the boundary of the Möbius strip.

For \( u = 0 \) we obtain a circle contained completely in the interior of \( M \). This is where the continuously extendability gets violated. The tangent vectors are

\[
X_u(u, v) = \begin{pmatrix} \cos v \cos(2v) \\ \cos v \sin(2v) \\ \sin v \end{pmatrix} \quad \text{and} \quad X_v(u, v) = \begin{pmatrix} -u \sin v \cos(2v) - 2(2 + u \cos v) \sin(2v) \\ -u \sin v \sin(2v) + 2(2 + u \cos v) \cos(2v) \\ u \cos v \end{pmatrix}.
\]

We want to find a contradiction for the continuously extendability of the unit normal vector to the circle \((u = 0)\). First, the parametrisation for \( v = 0 \) and \( v = \pi \).

\[
X(u, 0) = \begin{pmatrix} 2 + u \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad X(u, \pi) = \begin{pmatrix} 2 - u \\ 0 \\ 0 \end{pmatrix}
\]

For \( u \rightarrow 0 \) we have \( \lim_{u \rightarrow 0} X(u, 0) = \lim_{u \rightarrow 0} X(u, \pi) \).

What happens to the unit normal vector? Consider the tangential vectors for \( v = 0 \) and \( v = \pi \):

\[
X_u(u, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_v(u, 0) = \begin{pmatrix} 0 \\ 4 + 2u \\ u \end{pmatrix} \quad \text{and} \quad X_u(u, \pi) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad X_v(u, \pi) = \begin{pmatrix} 0 \\ 4 - 2u \\ -u \end{pmatrix}
\]

Taking the limit \( u \rightarrow 0 \) yields:

\[
\lim_{u \rightarrow 0} X_u(u, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lim_{u \rightarrow 0} X_v(u, 0) = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \quad \text{and} \quad \lim_{u \rightarrow 0} X_u(u, \pi) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \lim_{u \rightarrow 0} X_v(u, \pi) = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}
\]

In the first case \((v = 0)\) the normal vector is \((0, 0, 4)^T\) and the unit normal vector is \((0, 0, 1)^T\). In the second case \((v = \pi)\) the normal vector is \((0, 0, -4)^T\) and the unit normal vector is \((0, 0, -1)^T\). The two vectors point in different directions. Hence, the normal vector field is not continuously extendable.
The plots for $v \in [0, \pi/2], [0, \pi], [0, 3\pi/2], [0, 2\pi]$ and $u \in [0, 1]$ visualise how the Möbius strip is parametrised. You can also see in which direction the normal vector is pointing for different parameters $u$ and $v$. 😄