Keywords: separation of variables, probability, conditional probability

Exercise 1 (10 points)

Solve the linear partial differential equation of second order

\[ x^2 \frac{\partial^2 u}{\partial x^2}(x, y) = \frac{\partial u}{\partial y}(x, y) - x \frac{\partial u}{\partial x}(x, y), \quad x, y > 0, \]

by means of the method of separation of variables.

a) Determine all solutions of the form

\[ u(x, y) = v(x) w(y), \quad x, y > 0, \]

with functions \( v, w : \mathbb{R} \to \mathbb{R}. \)

b) Find a solution \( u \) such that it satisfies

\[ u(0, y) = 0 \quad \text{for} \quad y > 0 \quad \text{and} \quad u(x, 0) = x^2 \quad \text{for} \quad x > 0. \]

Proposal for a Solution to Exercise 1

a) The method of separation of variables leads to:

\[ x^2 v''(x)w(y) = v(x)w'(y) - xv'(x)w(y) \]

and

\[ \frac{x^2 v'(x) + xv'(x)}{v(x)} = \frac{w'(y)}{w(y)} = k^2 \]

with a constant \( k \in \mathbb{R}. \) The differential equation

\[ x^2 v''(x) + xv'(x) - k^2 v(x) = 0 \]

with \( x > 0 \) is an Eulerian differential equation. With the ansatz \( v(x) = x^\lambda \) we have

\[ \lambda(\lambda - 1)x^\lambda + \lambda x^\lambda - k^2 x^\lambda = 0, \]

i.e. the characteristic polynomial is \( \lambda^2 - k^2 = 0. \)

1. case \( k = 0: \) The general solution is

\[ v(x) = c_1 + c_2 \ln(x), \quad x > 0. \]
The differential equation for $w$ is $w'(x) = 0$ and has the solution $w(x) = 1$. All in all, we have

$$u(x, y) = c_1 + c_2 \ln(x)$$

with $c_1, c_2 \in \mathbb{R}$.

2. case $k \neq 0$: The general solution is

$$v(x) = c_1 x^k + c_2 x^{-k}, \quad x > 0$$

The differential equation for $w$ is $w'(y) = k^2 w(y)$ and has the solution $w(y) = e^{k^2 y}$. Thus,

$$u(x, y) = (c_1 x^k + c_2 x^{-k}) e^{k^2 y}, \quad x, y > 0.$$ 

b) Since the boundary conditions are at $x = 0$, solutions of the form $x^{-k}$ for $k > 0$ or of the form $\ln(x)$ are not feasible. Hence, we only consider

$$u(x, y) = c_1 x^k e^{k^2 y} + c_3, \quad x, y > 0.$$ 

From $u(0, y) = 0$ we have that $c_3 = 0$. With $c_1 = 1$ and $k = 2$ we get a solution:

$$u(x, y) = x^2 e^{4y}, \quad x, y > 0$$

of the differential equation which satisfies both conditions.
Exercise 2 (10 points)
Use the separation of variables ansatz \( u(x, y) = v(x) \cdot w(y) \) to determine solutions of the linear partial differential equation of second order

\[
2yu_{xx} - (1 + y^2)u_y + 4yu = 0.
\]

Proposal for a Solution to Exercise 2
Plugging in the ansatz in the differential equation yields

\[
2yv''w - (1 + y^2)v'w' + 4ywv = 0.
\]

For \( v, w, y \neq 0 \), we derive the condition

\[
\frac{v''}{v} + 2 = \frac{(1 + y^2)w'}{2yw} = k \quad \text{with constant } k.
\]

The left hand side only depends on \( x \), while the right hand side only depends on \( y \). Hence, both sides are constant and equal to a constant \( k \) and the problem decomposes into two ordinary differential equations:

\[
v'' + (k - 2)v = 0 \quad \text{and} \quad w' = \frac{2kyw}{1 + y^2}.
\]

The general solutions for \( k \neq 2 \) are

\[
v(x) = c_1 e^{\sqrt{k-2}x} + c_2 e^{-\sqrt{k-2}x} \quad \text{and} \quad w(y) = c_3 (1 + y^2)^k.
\]

This yields for \( k > 2 \) the solutions in product form

\[
u(x, y) = \left( ae^{\sqrt{k-2}x} + be^{-\sqrt{k-2}x} \right) (1 + y^2)^k \quad \text{with} \quad a, b \in \mathbb{R},
\]

and for \( k < 2 \) we get

\[
u(x, y) = \left( a \sin(\sqrt{2-k}x) + b \cos(\sqrt{2-k}x) \right) (1 + y^2)^k \quad \text{with} \quad a, b \in \mathbb{R}.
\]
Exercise 3  (10 points)
In order to improve its competencies in the domain of “finance”, the KIT decide to enter the market of gambling and offer a game of pure chance. A gambler earns one Euro if s/he wins. Otherwise s/he has to pay one Euro to the KIT.
The following rules apply: from three dice, the appearance of which will be specified soon, the gambler chooses the one s/he wants to play with; then the KIT choose one from the remaining two. The gambler and the KIT roll their respective dice simultaneously (once). The highest score wins.
The three dice have the following properties: **die 1** shows the digits 1,1,5,5,9,9; **die 2** the digits 2,2,6,6,7,7; **die 3** the digits 3,3,4,4,8,8.
"Do you want to try your luck? Do you want to choose any particular die to play with?"

Proposal for a Solution to Exercise 3
Whether you try your luck obviously depends on your gambler’s nature. However, playing the game is – from the monetary point of view – not a good idea since the KIT can gain an advantage by choosing the right die: On average it wins in 5 out of 9 games. There is no advantage in choosing any particular die.

We will show: If the player chooses die \( W_i \), then the KIT can choose another die \( W_j \) such that its winning probability is \( \frac{5}{9} \).

Case 1: The player chooses die 1, the KIT die 2.
There are the following equally probable results:
\[
\Omega = \{ (1,2), (1,6), (1,7), (5,2), (5,6), (5,7), (9,2), (9,6), (9,7) \}.
\]
The underlined results are those where the KIT is winning. Here \( \omega = (\omega_1, \omega_2) \in \Omega \) denotes the event, where the player roles \( \omega_1 \) and the KIT roles \( \omega_2 \). We see that the KIT has a probability of winning of \( \frac{5}{9} \).

Case 2: The player chooses die 2, the KIT die 3.
There are the following equally probable results:
\[
\Omega = \{ (2,3), (2,4), (2,8), (6,3), (6,4), (6,8), (7,3), (7,4), (7,8) \}.
\]
The underlined results are those where the KIT is winning. We see that the KIT has a probability of winning of \( \frac{5}{9} \).

Case 3: The player chooses die 3, the KIT die 1.
There are the following equally probable results:
\[
\Omega = \{ (3,1), (3,5), (3,9), (4,1), (4,5), (4,9), (8,1), (8,5), (8,9) \}.
\]
The underlined results are those where the KIT is winning. We see that the KIT has a probability of winning of \( \frac{5}{9} \).
Exercise 4 (10 points)

Suppose that a box contains six white and four black balls. Susan draws five balls from the box without placing them back.

a) Compute the probability that the first two balls are white.

b) Compute the probability that Susan draws exactly four white balls in total.

c) Susan makes the following statement: “I have drawn exactly four white balls.” What is the (conditional) probability that the first two balls were white?

Proposal for a Solution to Exercise 4

Let \( G = \{1, \ldots, 10\} \) be the set of the then balls and \( W = \{1, \ldots, 6\} \subset G \) the set of the six white balls. We denote the ball drawn by \( \omega_i \in G, \ i \in \{1, \ldots, 5\} \).

a) It suffices to look at the first two balls. We have permutations without repetition. Thus the probability space is

\[ \Omega = \{ \omega = (\omega_1, \omega_2) : \omega_1, \omega_2 \in G \text{ and } \omega_1 \neq \omega_2 \}. \]

We have \( \text{card}(\Omega) = 10 \cdot 9 = 90 \). Every result \( \omega \in \Omega \) has the same probability, i.e. we have a Laplace experiment. We are interested in the probability that \( \omega \) is an element of the set

\[ A = \{ \omega \in \Omega : \omega_1, \omega_2 \in W \} \]

We have \( \text{card}(A) = 6 \cdot 5 = 30 \). Hence, the probability we are looking for is

\[ P(A) = \frac{\text{card}(A)}{\text{card}(\Omega)} = \frac{30}{90} = \frac{1}{3}. \]

b) Now we are only interested in the number of drawn white balls but not in the order they are drawn, i.e. we have combinations without repetitions. We sort the drawn balls by their number, which yields the following probability space:

\[ \Omega = \{ \omega = (\omega_1, \ldots, \omega_5) : \omega_j \in G \text{ and } \omega_1 < \cdots < \omega_5 \} \]

Each result \( \omega \in \Omega \) has the same probability, i.e. we have a Laplace experiment, this time with \( \text{card}(\Omega) = \binom{10}{5} \). The event we are looking for is

\[ B = \{ \omega \in \Omega : \omega_1, \ldots, \omega_4 \in W \text{ and } \omega_5 \in G \setminus W \}. \]

We want to draw four out of six white balls and one out of four black balls. Thus we have as the number of possible results \( \text{card}(B) = \binom{6}{4} \cdot \binom{4}{1} \). Hence, the probability we are looking for is

\[ P(B) = \frac{\text{card}(B)}{\text{card}(\Omega)} = \frac{\binom{6}{4} \cdot \binom{4}{1}}{\binom{10}{5}} = \frac{6! \cdot 4!}{4! \cdot 2! \cdot 10! \cdot 5!} = \frac{5^2 \cdot 7 \cdot 5^2 \cdot 2^2 \cdot 2^2}{2^2 \cdot 3 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = \frac{5}{73} = \frac{5}{21}. \]

c) We are interested in the conditional probability \( P(A \mid B) \). Formally we should define both events in the same probability space \( \Omega \). However, we simplified that to make the calculations in a) and b) easier.
We can calculate the conditional probability by means of the Bayes formula and reduce it to calculating $P(B \mid A)$. The latter is easier to calculate: If we already know that the first two balls are white, then in the third draw we have four white and four black balls in the box. Under this conditions we have that the probability to draw four white balls is the same as with the beginning of the third draw to draw two white balls and one black ball. We denote this event with $C$ and its probability can be calculated similar to what we did in b):

$$P(B \mid A) = P(C) = \frac{\binom{4}{2} \cdot \binom{4}{1}}{\binom{8}{3}} = \frac{\frac{4!}{2! \cdot 1!}}{\frac{8!}{3! \cdot 5!}} = \frac{3 \cdot 2 \cdot 4}{8 \cdot 7} = \frac{3}{7}.$$

Using Bayes formula we have

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} = \frac{\frac{3}{7} \cdot \frac{1}{3}}{\frac{5}{21}} = \frac{3}{5}.$$
Exercise 5  (10 points)

Mr A is looking for a place to park his car. To this end, he continuous to drive around a block of flats in front of which there are three parking spaces. From experience Mr A knows that the probability of finding any of no, one, two or all three parking space free is the same on his first tour around the block. During every further tour around the block, the probability of finding no space free is halved while the probability of finding at least one space free increases accordingly.

Mr A drives around the block for as long as it takes to find a place to park, but no more than three times.

a) Determine the sample space for this random experiment.

b) Draw a tree diagram for this random experiment from which the probability of Mr A finding a parking space after at most three tours around the block can be found.

c) What is the probability of Mr A finding a place to park on first tour around the block on at least three out of five days?

Proposal for a Solution to Exercise 5

a) There are many possibilities for the sample space. The simplest one being

\[ \Omega = \{ "'Mr A finds a parking spot'", "'Mr A finds no parking spot'" \} . \]

A slightly more accurate variant would be

\[ \Omega = \{ "'Mr A finds a parking spot on the first tour around'" ,
          "'Mr A finds a parking spot on the second tour around'" ,
          "'Mr A finds a parking spot on the third tour around'" ,
          "'Mr A finds no parking spot'" \} . \]

It can even get more complicated (and precise): Let \((a, b, c)\) be the event where on the first tour around the block we have \(a\) free parking spot, on the second tour around we have \(b\) free spots and on the third tour around we have \(c\) free parking spots. If \(a > 0\), then there will be no second (or third) tour around the block and we write \((a)\). Similarly we write \((a, b)\) if \(b > 0\). This way we can describe the sample space as

\[ \Omega = \{ (3), (2), (1), (0, 3), (0, 2), (0, 1), (0, 0, 3), (0, 0, 2), (0, 0, 1), (0, 0, 0) \} . \]

Note: If you define a sample space in this form, you must define what the numbers mean!

b) The tree diagram looks as follows:
From the diagram we can read of the probability we are looking for:

\[
p = \frac{3}{4} + \frac{1}{4} \cdot \frac{7}{8} + \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{15}{16} = \frac{3 \cdot 128 + 7 \cdot 16 + 15}{512} = \frac{511}{512}
\]

Alternatively, we can use the following calculation:

\[
p = 1 - \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{1}{16} = 1 - \frac{1}{512} = \frac{511}{512}
\]

c) The probabilities to find on the first tour around the block none, one, two or three free parking spots are equal on every day. Thus, we have

\[
p(\text{"no free parking spot on the first tour around the block on at least } 3 \text{ out of } 5 \text{ days"}) = \frac{\text{number of favorable events}}{\text{number of possible events}}.
\]

The elementary results are the elements of \(\{0, 1, 2, 3\}^5\): At day \(n\) with \(n = 1, \ldots, 5\) are none, one, two, or three parking spots free. Hence, the number of possible events is \(4^5 = 1024\). Let \((a_1, a_2, a_3, a_4, a_5) \in \{0, 1, 2, 3\}^5\), where \(a_n\) for \(n = 1, \ldots, 5\) means that at the \(n\)-th day there were \(a_n\) free parking spots on the first tour around the block.

For a favourable event ("no free parking spot on at least three days") there must be exactly three, exactly four or exactly five numbers \(a_n\) be equal to zero, for all the other numbers there are 9, 3, or 1 possibilities, respectively. Hence, we have

\[
\text{Number of favorable events} = \binom{5}{3} \cdot 9 + \binom{5}{4} \cdot 3 + \binom{5}{5} = 90 + 15 + 1 = 106.
\]

Thus,

\[
p(\text{"no free parking spot on the first tour around the block on at least } 3 \text{ out of } 5 \text{ days"}) = \frac{106}{4^5} = \frac{53}{512}.
\]