Advanced Mathematics III
Exercise Sheet 13

Keywords: Bernoulli experiment, expectation value, variance, distribution function

Exercise 1 (10 points)
Consider a multiple choice test consisting of 30 problems with 4 possible responses each. For every problem, let only one of these responses offer a correct solution. Moreover, assume that a respondent selects all responses (only one per problem) independently of one another. Calculate the probabilities that, by randomly selecting responses, in total
a) no problem,
b) exactly one problem,
c) more than 15 problems,
d) between 7 and 9 problems, and
e) all problems will be solved correctly?

Exercise 2 (10 points)
Let \( \alpha, \beta \in (0, \infty) \). A real-valued function \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by
\[
  f(x) = \begin{cases} \frac{\alpha}{\beta} \left( \frac{\beta}{x} \right)^{\alpha+1}, & x \geq \beta, \\ 0, & \text{otherwise}. \end{cases}
\]
(a) Show that \( f \) constitutes a probability density function (pdf).
In the following, let \( X \) be a random variable which is continuously distributed with probability density \( f \).
(b) Determine the cumulative distribution function of \( X \).
(c) Find conditions for \( \alpha \) and \( \beta \) under which the expectation value of \( X \) does exist. Explicitly compute the expectation value in this case.
(d) Find conditions for \( \alpha \) and \( \beta \) under which the variance of \( X \) does exist. Explicitly compute the variance in this case.

Exercise 3 (10 points)
The dial of a stop watch has ticks every 0.2s. What is the probability that we measure the time with an error greater than 0.05? (Here the measurement is taken by rounding up to the next tick.)
Exercise 4 (10 points)

A jury consists of three people: two of which decide independently of each other and correctly with probability $p$, the third member uses a coin flip for the decision. The jury’s overall vote is decided by majority rule.

A second jury only consists of one person which decides correctly with probability $p$.

Which of the two juries has a higher probability to make the correct decision?

Exercise 5 (10 points)

A delivery of 25 machines, which can only be distinguished by their serial numbers, contains 4 machines that are defective. Selecting $k$ of these machines is being referred to as drawing a sample of size $k$ from the delivery, without repetition.

(a) How many different samples of size 5 can be drawn?

(b) What is the probability that a sample of size 5 contains

   (i) exactly two defective,       (iii) only non-defective,
   (ii) at most one defective,     (iv) at least one defective machine(s).

(c) How do these probabilities change if each of the 5 machines is first being selected, then checked, and then put back, i.e., a repetition of selection becomes possible?

(d) Which of the random experiments (b) and (c) constitutes a Bernoulli experiment?