Advanced Mathematics III
Exercise Sheet 13

Keywords: Bernoulli experiment, expectation value, variance, distribution function

Exercise 1 (10 points)
Consider a multiple choice test consisting of 30 problems with 4 possible responses each. For every problem, let only one of these responses offer a correct solution. Moreover, assume that a respondent selects all responses (only one per problem) independently of one another. Calculate the probabilities that, by randomly selecting responses, in total

a) no problem,
b) exactly one problem,
c) more than 15 problems,
d) between 7 and 9 problems, and
e) all problems will be solved correctly?

Proposal for a Solution to Exercise 1
Solving the test by guessing is a Bernoulli experiment with the single experiment being "guess the correct answer out of four possible ones". The single experiment is repeated \( n = 30 \) times and the the probability of success is \( p = \frac{1}{4} \). Let the random variable \( X \) describe the number of correctly solved problems.

a) We are looking for the probability

\[
P(X = 0) = \binom{30}{0} \cdot \left( \frac{1}{4} \right)^0 \cdot \left( 1 - \frac{1}{4} \right)^{30} = \binom{30}{0} \cdot \left( \frac{1}{4} \right)^0 \cdot \left( \frac{3}{4} \right)^{30} = \left( \frac{3}{4} \right)^{30} \approx 0,02\%.
\]

b) We are looking for the probability

\[
P(X = 1) = \binom{30}{1} \cdot \left( \frac{1}{4} \right)^1 \cdot \left( \frac{3}{4} \right)^{29} \approx 0,2\%.
\]

c) We are looking for the probability

\[
P(X > 15) = 1 - P(X \leq 15) = 1 - \sum_{k=0}^{15} \binom{30}{k} \cdot \left( \frac{1}{4} \right)^k \cdot \left( \frac{3}{4} \right)^{30-k} \approx 0,08\%.
\]
d) We are looking for the probability

\[ P(6 < X \leq 9) = P(X \leq 9) - P(X \leq 6) \]
\[ = \sum_{k=0}^{9} \binom{30}{k} \cdot \left(\frac{3}{4}\right)^{30-k} - \sum_{k=0}^{6} \binom{30}{k} \cdot \left(\frac{1}{4}\right)^{k} \cdot \left(\frac{3}{4}\right)^{30-k} \]
\[ \approx 45.5\% . \]

e) We are looking for the probability

\[ P(X = 30) = \binom{30}{30} \cdot \left(\frac{1}{4}\right)^{30} \cdot \left(\frac{3}{4}\right)^{0} \approx 10^{-18}\% . \]
Exercise 2 (10 points)

Let $\alpha, \beta \in (0, \infty)$. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{\alpha}{\beta} \left(\frac{\beta}{x}\right)^{\alpha+1}, & x \geq \beta, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Show that $f$ constitutes a probability density function (pdf).

In the following, let $X$ be a random variable which is continuously distributed with probability density $f$.

(b) Determine the cumulative distribution function of $X$.

(c) Find conditions for $\alpha$ and $\beta$ under which the expectation value of $X$ does exist. Explicitly compute the expectation value in this case.

(d) Find conditions for $\alpha$ and $\beta$ under which the variance of $X$ does exist. Explicitly compute the variance in this case.

Proposal for a Solution to Exercise 2

a) Obviously $f(x) \geq 0$ for all $x \in \mathbb{R}$. Furthermore, we have

$$\int_{-\infty}^{\infty} f(x) \, dx = \frac{\alpha}{\beta} \int_{\beta}^{\infty} \left(\frac{\beta}{x}\right)^{\alpha+1} \, dx = \alpha \beta^\alpha \int_{\beta}^{\infty} x^{-\alpha-1} \, dx = \frac{\alpha \beta^\alpha}{1 - \alpha} \left[ x^{-\alpha} \right]_{\beta}^{\infty} = 1.$$

Thus, $f$ is a probability density.

b) The cumulative distribution function of $X$ is:

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} f(t) \, dt = \begin{cases} 0, & \text{if } x \leq \beta, \\ 1 - \left(\frac{\beta}{x}\right)^{\alpha}, & \text{for } x > \beta. \end{cases}$$

c) The expectation value of $X$ is, if it exists, given by

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx = \frac{\alpha}{\beta} \int_{\beta}^{\infty} x \cdot \left(\frac{\beta}{x}\right)^{\alpha+1} \, dx = \alpha \beta^\alpha \int_{\beta}^{\infty} x^{-\alpha} \, dx = \alpha \beta^\alpha \left[ \frac{1}{1 - \alpha} x^{1-\alpha} \right]_{\beta}^{\infty}.$$

For $\alpha > 1$:

$$E(X) = -\alpha \beta^\alpha \frac{1}{1 - \alpha} \beta^{1-\alpha} = \frac{\alpha \beta}{\alpha - 1}.$$

Hence, the expectation value exists only if $\alpha > 1$. In this case it is $\frac{\alpha \beta}{\alpha - 1}$. 

3
d) By definition of the variance, we need the existence of the expectation value of $X$. Hence, we must have $\alpha > 1$. Also by definition, the variance is calculated as

\[
\text{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 \cdot f(x) \, dx = \int_{-\infty}^{\infty} (x^2 - 2E(X)x + E(X)^2) \cdot f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx - 2E(X) \cdot \int_{-\infty}^{\infty} x \cdot f(x) \, dx + E(X)^2 \cdot \int_{-\infty}^{\infty} f(x) \, dx = E(X), \text{ exists by assumption}
\]

\[
= \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx - E(X)^2.
\]

Hence, for the variance to exist, we must also have that the integral $\int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx$ exists.

We have

\[
\int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx = \int_{\beta}^{\infty} x^2 \cdot \frac{\beta^{\alpha+1}}{\beta} \left(\frac{\beta}{x}\right)^{\alpha} \, dx = \alpha \beta^\alpha \int_{\beta}^{\infty} x^{1-\alpha} \, dx = \alpha \beta^\alpha \left[ \frac{1}{2-\alpha} x^{2-\alpha} \right]_{\beta}^{\infty}
\]

\[
= \frac{\alpha \beta^\alpha}{\alpha - 2} \frac{1}{\beta^{2-\alpha}} = \frac{\alpha \beta^2}{\alpha - 2}.
\]

Therefore, the variance exists in the case $\alpha > 2$ and we have:

\[
\text{Var}(X) = \frac{\alpha \beta^2}{\alpha - 2} - \frac{\alpha^2 \beta^2}{(\alpha - 1)^2} = \frac{\alpha \beta^2 ((\alpha - 1)^2 - \alpha(\alpha - 2))}{(\alpha - 1)^2(\alpha - 2)} = \frac{\alpha \beta^2}{(\alpha - 1)^2(\alpha - 2)}.
\]
Exercise 3 (10 points)
The dial of a stop watch has ticks every 0.2s. What is the probability that we measure the time with an error greater than 0.05? (Here the measurement is taken by rounding up to the next tick.)

Proposal for a Solution to Exercise 3
Let $X$ be the random variable which describes the event that the stop watch was stopped in the time interval $[0, 0.2]$. Since the stop watch can be activated with equal probability in this time interval, we have that $X$ is uniformly distributed on $[0, 0.2]$. The distribution function of $X$ is:

$$F(x) = P(X < x) = \begin{cases} 
0, & x < 0, \\
\frac{1}{0.2}x, & 0 \leq x \leq 0.2, \\
1, & x \geq 0.2. 
\end{cases}$$

By the problem description we have that the error is greater that 0.05, if $0.05 < X < 0.15$. The probability for this is:

$$P(0.05 < X < 0.15) = P(X < 0.15) - P(X < 0.05) = F(X < 0.15) - F(X < 0.05) = \frac{1}{0.2} (0.15 - 0.05) = 0.5.$$
Exercise 4 (10 points)

A jury consists of three people: two of which decide independently of each other and correctly with probability $p$, the third member uses a coin flip for the decision. The jury’s overall vote is decided by majority rule.

A second jury only consists of one person which decides correctly with probability $p$.

Which of the two juries has a higher probability to make the correct decision?

Proposal for a Solution to Exercise 4

For the jury with three people: let $A$ be the event that the first jury member has made the correct decision, let $B$ be the event that the second jury member has made the correct decision, and let $C$ be the event that the third jury member has made the correct decision.

By the description of the jury we have

$$P(A) = P(B) = p \quad \text{and} \quad P(C) = \frac{1}{2}.$$

We look for $P(S)$ where

$$S = (A \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C).$$

Since the events $R_1$, $R_2$, $R_3$, $R_4$ are pairwise disjoint, we have

$$P(S) = P(R_1 \cap R_2 \cap R_3 \cap R_4) = P(A \cap B \cap C) + P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$$

stochastically independent

$$= P(A)P(B)P(C) + P(A)P(B)P(C^c) + P(A)P(B^c)P(C) + P(A^c)P(B)P(C)$$

$$= \frac{1}{2}p^2 + \frac{1}{2}p^2 + p(1 - p)\frac{1}{2} + p(1 - p)\frac{1}{2}$$

$$= p^2 + p(1 - p) = p.$$

Hence, both described juries have the same probability to decide correctly.
Exercise 5 (10 points)

A delivery of 25 machines, which can only be distinguished by their serial numbers, contains 4 machines that are defective. Selecting \( k \) of these machines is being referred to as drawing a sample of size \( k \) from the delivery, without repetition.

(a) How many different samples of size 5 can be drawn?

(b) What is the probability that a sample of size 5 contains

(i) exactly two defective, 
(ii) at most one defective, 
(iii) only non-defective, 
(iv) at least one defective machine(s).

(c) How do these probabilities change if each of the 5 machines is first being selected, then checked, and then put back, i.e., a repetition of selection becomes possible?

(d) Which of the random experiments (b) and (c) constitutes a Bernoulli experiment?

Proposal for a Solution to Exercise 5

(a) Because the order of the machines does not matter, there exists \( \binom{25}{5} \) = 53130 possibilities to draw samples of size 5.

(b) (i) There exists \( \binom{21}{2} \cdot \binom{4}{3} \cdot \binom{4}{1} \) = 7980 samples, which have exactly two defective machines. hence, the probability is \( p_2 = 7980/53130 = 38/253 \approx 0,150 \).

(ii) \( \binom{21}{2} \cdot \binom{4}{0} + \binom{21}{1} \cdot \binom{4}{1} \) = 44289 samples with 0 or 1 defective machines. Hence the probability is \( p_0 + p_1 = 44289/53130 = 2109/2530 \approx 0,834 \)

(iii) \( \binom{21}{0} \cdot \binom{4}{0} \) = 20349 samples with 0 defective machines. Hence the probability is \( p_0 = 20349/53130 = 969/2530 \approx 0,383 \)

(iv) With the result of (iii) we deduce that there exists 53130 – 20349 = 32781 samples with at least one defective machine. Hence the probability is \( 1 - p_0 = 1 - 969/2530 = 1561/2530 \approx 0,617 \)

(c) In the case that the machines are drawn and put back, we have a Bernoulli experiment with probability of success \( p = \frac{4}{25} = 0,16 \).

(i) \( p_2 = \binom{5}{2} \cdot \left( \frac{4}{25} \right)^2 \cdot \left( \frac{21}{25} \right)^3 \approx 0,152 \)

(ii) \( p_0 + p_1 = \left[ \binom{5}{0} \cdot 21 + \binom{5}{1} \cdot 4 \right] \cdot \frac{21^4}{25^5} \approx 0,817 \)

(iii) \( p_0 = \binom{5}{0} \cdot \frac{21^5}{25^5} \approx 0,418 \)

(iv) \( 1 - p_0 \approx 0,582 \)

(d) The experiment in part (c) is a Bernoulli experiment, the one in part (b) is not.

Remark: In part (c) is the probability of the event to draw 5 defective machines positive: \( \binom{5}{5} \cdot \frac{4^5}{25^5} \approx 1 \cdot 10^{-4} \). In part (b) is the event impossible, i.e. has probability 0.

♦