Hyperuniformity of inflation tilings

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Previously: on inflation tilings...

- Inflation tilings: “self-similar” structure
- Characterized by map $\varrho : \Omega_T \to \Omega_T$
- Often “more structured” than cut-and-project
  - Hyperuniform with high decay, given by λ_2
  - Goal of today: prove this! (in the Fibonacci case)

What is renormalization, really?

Theorem

Let T_o be a nice inflation tiling. Let μ be the invariant probability measure on Ω_{T_o}. and $\varrho_*\mu$ be defined by $\varrho_*\mu(B) = \mu(\varrho(B))$. Then $\mu = \varrho_*\mu$.

Proof.

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- Unique ergocity $\implies \mu = \varrho_*\mu$ up to a constant
- $\varrho(\Omega_{T_o}) = \Omega_{T_o}$ (minimality)

□

Renormalization for colored point processes

Theorem

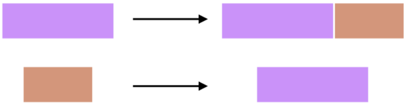
Let T_o be a nice inflation tiling with colored point process $\mathbf{\Lambda}$. Then

$$\mathbf{\Lambda} \sim \varrho(\mathbf{\Lambda}) = R_\varrho D_\lambda \mathbf{\Lambda}$$

Fibonacci tilings

The Fibonacci inflation rule

- Prototiles α, β
  - Lengths τ and 1
- Scaling constant: $\lambda = \tau = \frac{1+\sqrt{5}}{2}$
- Inflation rule ϱ



Fibonacci tilings

The Fibonacci inflation matrix

The inflation matrix of the Fibonacci inflation is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- $\lambda_{PF} = \tau$, $v_{PF} = (\tau^{-1}, \tau^{-2})^T$
- $\lambda_2 = -\tau^{-1}$

Renormalization for Fibonacci processes

Theorem

Let $\boldsymbol{\Lambda} = (\Lambda_\alpha, \Lambda_\beta)$ *be the Fibonacci colored process. Then, as measures:*

$$\begin{aligned} \Lambda_\alpha &\sim T_{-\frac{1}{2}} D_\tau \Lambda_\alpha + D_\tau \Lambda_\beta \\ \Lambda_\beta &\sim T_{\frac{\tau-1}{2}} D_\tau \Lambda_\alpha \end{aligned} \tag{1}$$

Relative Fibonacci frequencies

Theorem

The relative frequencies of the prototiles in Fibonacci tilings are:
$\text{freq}(\alpha) = \tau^{-1}$, $\text{freq}(\beta) = \tau^{-2}$

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- For Borel B: $\mathbb{E}[\Lambda_i(B)] = \iota(\Lambda_i) m_{\mathbb{R}}(B)$

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- For Borel B: $\mathbb{E}[\Lambda_i(B)] = \iota(\Lambda_i) m_{\mathbb{R}}(B)$
- Use renormalization!

$$\begin{aligned} \iota(\Lambda_\alpha) =& \tau^{-1}(\iota(\Lambda_\alpha) + \iota(\Lambda_\beta)) \\ \iota(\Lambda_\beta) =& \tau^{-1}\iota(\Lambda_\alpha) \end{aligned} \tag{2}$$

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- Frequencies: $v = \tau^{-1} A v$ with $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

□

Reminder: matrix autocorrelation and diffraction

Definition

Autocorrelation matrix $\boldsymbol{\eta} = (\eta_{ij})_{i,j=1}^{\ell}$ given by

$$\eta_{ij}(f^* * g) = \mathbb{E}[\overline{\sum_{x \in \Lambda_i} f(x)} \sum_{y \in \Lambda_j} g(y)]$$

for all $f, g \in \mathcal{L}_c^{\infty}(G)$

- Define (spherical) diffraction componentwise

Renormalization for the Fibonacci diffraction matrix

We write $\hat{\boldsymbol{\eta}}(x) = \hat{\boldsymbol{\eta}}(x)$.

Theorem

The diffraction matrix of the Fibonacci colored process satisfies:

$$\hat{\boldsymbol{\eta}}(\tau^{-1}x) = \tau^{-2} A(x) \hat{\boldsymbol{\eta}}(x) A^{\dagger}(x)$$

where $A(x) = \begin{pmatrix} e^{-\tau\pi i x} & 1 \\ e^{\pi i x} & 0 \end{pmatrix}$

Question: how does $\hat{\boldsymbol{\eta}}(x)$ behave around 0?

Diffraction around 0 and spectral cocycles

Lemma (Fourier-Bohr coefficients)

There exists a vector-valued function such that

$$\hat{\boldsymbol{\eta}}(x) = v(x)v(x)^\dagger$$

and

$$\tau^{-1}A(x)v(x) = v(\tau^{-1}x)$$

Lemma

We have $v(\tau^{-k}x) = \tau^{-k}A^{(k)}(x)v(x)$, *where*

$$A^{(k)}(x) = A(\tau^{-k+1}x)\cdots A(x)$$

is the spectral cocycle associated to the diffraction.

Note: $A(0) = A$ is the inflation matrix.

Perron-Frobenius theory

Theorem (Perron-Frobenius, symmetric case)

Let $A \in \mathbb{R}^{\ell \times \ell}$ be a primitive, nonnegative matrix. Then:

- *A has positive maximal eigenvalue λ_{PF} with unique positive eigenvector v_{PF}*
- *If $v = v^{\parallel} + v^{\perp}$ is the orthogonal decomposition associated to v_{PF}, then:*

$$\|A^k v - \lambda_{PF}^k v^{\parallel}\| \leq \lambda_2^k v^{\perp}$$

Hyperuniformity of Fibonacci processes

Theorem

We have $v(\lambda^{-k}) = O_{k\to\infty}(\tau^{-2} + \epsilon)^k)$ *for all* $\epsilon > 0$

Sketch.

□

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We have $v(\lambda^{-k}) = O_{k\to\infty}((\tau^{-2}+\epsilon)^k)$ *for all* $\epsilon > 0$

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- For small x, $A^{(k)}(x) \approx A^k$

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- For small x, $A^{(k)}(x) \approx A^k$
- If $v(\tau^{-k})$ had some nontrivial v_{PF} component for all k, we would have

$$v(\tau^{-k}) = \tau^{-k} A^{(k)}(1) v(1) \gg 0$$

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- But $\hat{\boldsymbol{\eta}}$ locally finite $\implies \sum \|v(\tau^{-k})\|^2 < \infty$, so this is not possible!
- Therefore

$$v(\tau^{-k}) = \tau^{-k} A^{(k)} v(1) \lessapprox (\frac{\lambda_2}{\lambda_{PF}})^k = \tau^{-2k}$$

□

Hyperuniformity of Fibonacci processes

Corollary

We have

$$\hat{\boldsymbol{\eta}}(B_r^{\times}) = O(r^{4-\epsilon})$$

for $r \to 0$*, all* $\epsilon > 0$*.*
Therefore, Fibonacci tilings are hyperuniform ($o(r)$*) and number rigid (*$o(r^{2+\epsilon})$*)*

General case

Theorem (R., Baake-Grimm-Gähler)

Let T be a nice inflation tiling with colored point process $\boldsymbol{\Lambda}$. Assume it has pure point (spherical) diffraction $\hat{\boldsymbol{\eta}}$. Let λ be the scaling constant and $\rho_2(A_{red})$ be the second largest eigenvalue of the inflation matrix. Then

$$\hat{\eta}(B_r^{\times}) = O(r^{d+\alpha-\epsilon})$$

where

$$\alpha := d - 2d \frac{\log |\rho_2(A_{red})|}{\log \lambda^d}$$

for $r \to 0$, all $\epsilon > 0$.