Hyperuniformity of inflation tilings

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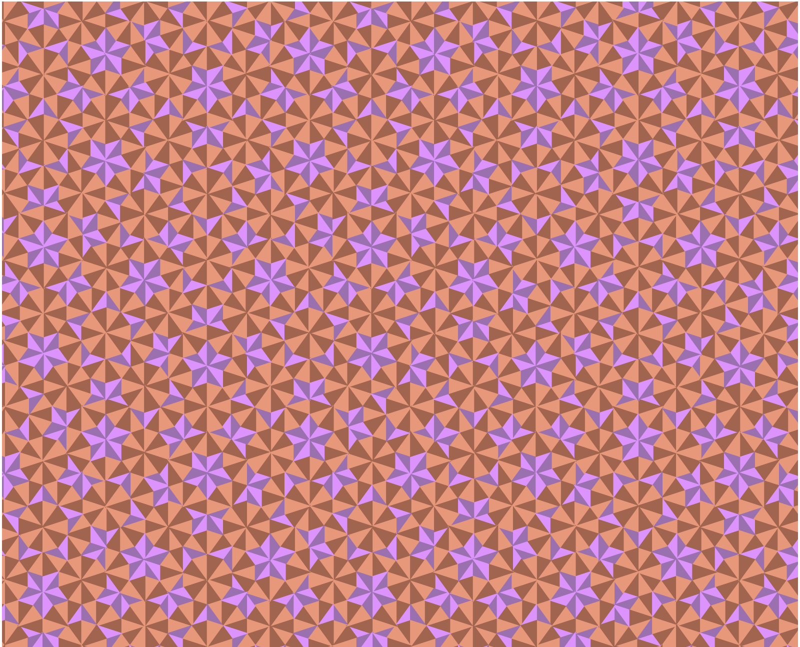
Section 1

Inflation tilings

A Penrose tiling



Inflation tilings

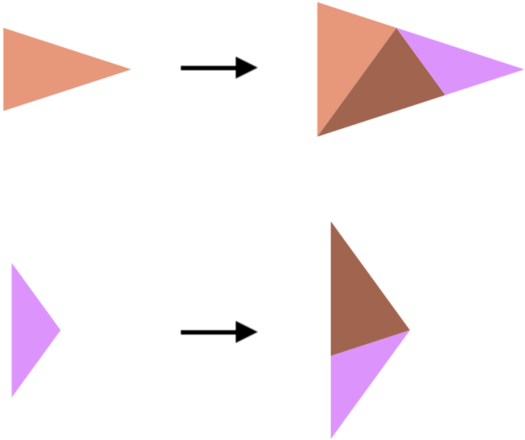
- Tilings with “self-similar” structure
- Penrose, Ammann-Beenker, Fibonacci
- Often overlap with cut-and-project

Settings

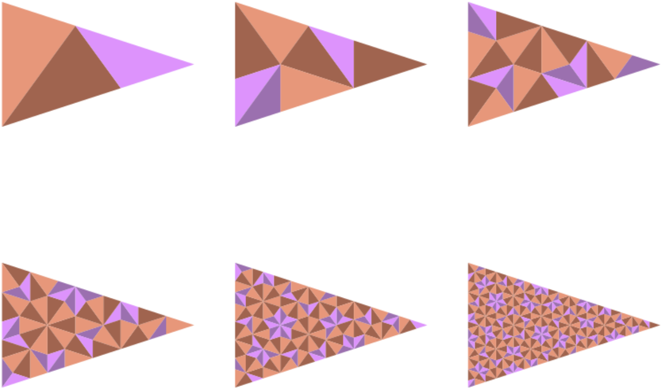
- Abelian case: $G = \mathbb{R}^d$, classical diffraction
- Euclidean case: $G = \mathbb{R}^d \rtimes O(d)$, $K = O(d)$, spherical diffraction
- We need *dilation structure* $(D_\lambda : G \to G)_{\lambda \in \mathbb{R}}$

The Penrose inflation rule

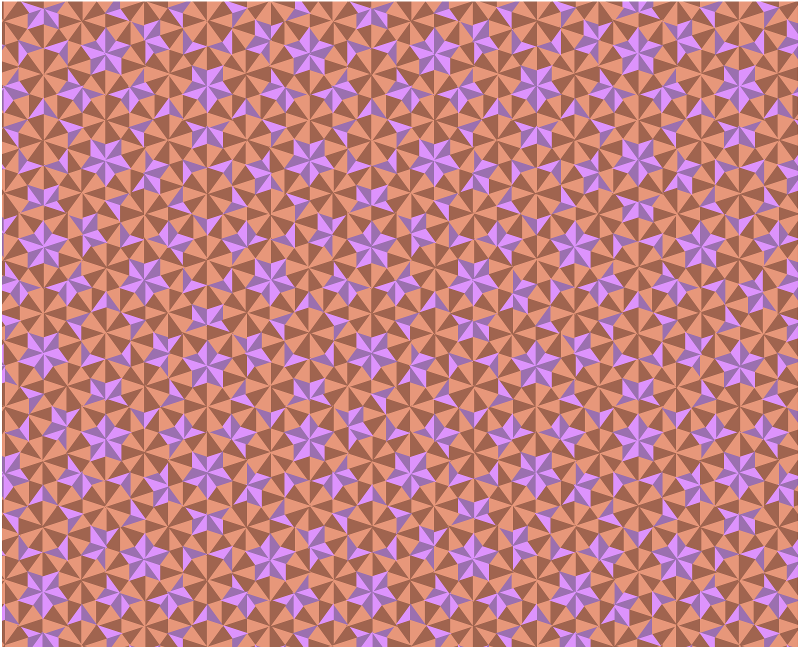
- Prototiles $\subset G/K$
  - half-kite, half-dart
- Scaling constant: $\lambda = \tau = \frac{1+\sqrt{5}}{2}$
- Inflation rule ϱ



The Penrose inflation rule



The Penrose inflation rule



Inflation matrices

Definition

The *inflation matrix* of the rule ϱ with prototiles $t_1, \cdots, t_\ell$ is defined by

$$A_{ij} = \#\{\text{appearances of } t_i \text{ in } \varrho(t_j)\}$$

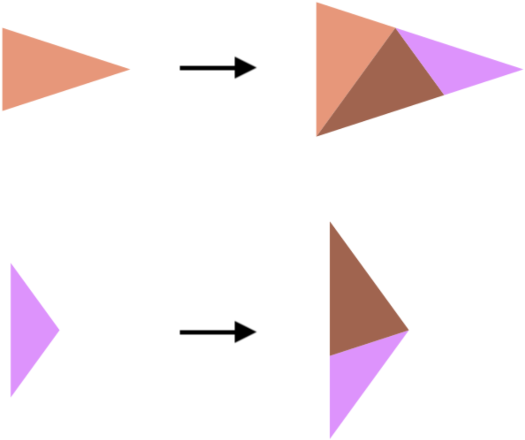
- Primitive: $A^k > 0$ for some k
- First eigenvector v_{PF}: relative frequencies
- Second eigenvalue: relevant to hyperuniformity

The Penrose inflation matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$v_{PF} = \begin{pmatrix} \tau^{-1} \\ \tau^{-2} \end{pmatrix}$$

$$v_{PF} \approx \begin{pmatrix} 0,618 \\ 0,382 \end{pmatrix}$$



Subtlety: Inflation tilings and groups

$G = \mathbb{R}^d$	$G = Iso(\mathbb{R}^d)$
40 prototiles	2 prototiles
$A_{\text{full}} \in \mathbb{R}^{40\times 40}$	$A_{\text{red}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
$Spec(A_{\text{full}}) = \{\tau^2, \tau, \cdots\}$	$Spec(A_{\text{red}}) = \{\tau^2, \tau^{-2}\}$
Abelian	Nonabelian
Classical diffraction	Spherical diffraction

Tiling space

- "Nice" inflation tilings T_o: Ω_{T_o} is compact, minimal and uniquely ergodic (over $\mathbb{R}^d$ or $\mathbb{R}^d \rtimes O(d)$)
- Nice: FLC and primitive
- Inflation map: continuous surjection $\varrho : \Omega_{T_o} \to \Omega_{T_o}$ with $\varrho(gT) = (D_\lambda g)\varrho(T)$

Inflation tilings and point processes

- We want point processes, not tilings!
- We want to keep track of the different kinds of tiles
- We want to work in the group $G = \mathbb{R}^d \rtimes O(d)$

Inflation tilings and point processes

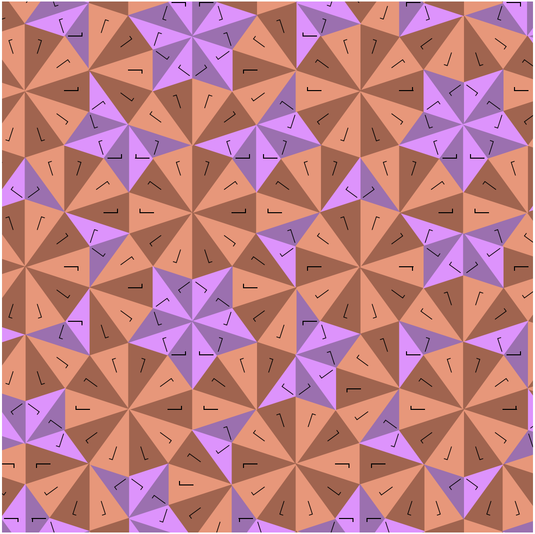
Example

- T_o: nice tiling of $\mathbb{R}^d$
- Prototiles: $t_1, \cdots, t_\ell \subset \mathbb{R}^d$

$$\Lambda_i \quad : \quad \Omega_{T_o} \to Cl(G)$$
$$T \mapsto \{g \in G \mid gt_i \in T\}$$

$(i \in [\ell])$

Inflation tilings and point processes



Colored point processes

Definition

A *colored point process* $\mathbf{\Lambda} = (\Lambda_1, \cdots, \Lambda_\ell)$ in a group G is a vector of point processes over the same probability space.

- Random tilings $\leftrightarrow$ colored point processes
- Total point process $\Lambda = \dot{\bigcup}_{i=1}^{\ell} \Lambda_i$

Reminder: autocorrelation of point processes

Definition

Autocorrelation: unique $\eta \in M(G)$ such that

$$\eta(f^* * g) = \mathbb{E}[\overline{\sum_{x\in\Lambda} f(x)} \sum_{y\in\Lambda} g(y)]$$

for $f, g \in \mathcal{L}_c^\infty(G)$

- $G = \mathbb{R}^d$: η is positive, pd $\implies$ (uncentered!) diffraction $\hat{\eta}$ exists and is positive
- $G = \mathbb{R}^d \rtimes O(d)$: spherical diffraction $\hat{\eta}$ exists

Matrix autocorrelation and diffraction

Definition

Autocorrelation matrix $\boldsymbol{\eta} = (\eta_{ij})_{i,j=1}^{\ell}$ given by

$$\eta_{ij}(f^* * g) = \mathbb{E}[\overline{\sum_{x \in \Lambda_i} f(x)} \sum_{y \in \Lambda_j} g(y)]$$

for all $f, g \in \mathcal{L}_c^{\infty}(G)$

- Define (spherical) diffraction componentwise

Matrix and total diffraction/autocorrelation

Lemma

Let $\boldsymbol{\eta} = (\eta_{ij})_{i,j=1}^{\ell}$ be the autocorrelation matrix and η the autocorrelation of the total point process Λ

- $\eta = \sum_{i,j=1}^{\ell} \eta_{ij}$
- $\hat{\eta} = \sum_{i,j=1}^{\ell} \hat{eta}_{ij}$
- $\eta \leq \|\boldsymbol{\eta}\| = \sum_{i,j=1}^{\ell} |\eta_{ij}|$

Section 2

Hyperuniformity

Reminder: hyperuniformity

Definition

A point process in $\mathbb{R}^d$ or $\mathbb{R}^d \rtimes O(d)$ is hyperuniform if

$$\mathrm{Var}(\#\Lambda \cap B_R) = o_{R\to\infty}(R^d)$$

Theorem

A point process in $\mathbb{R}^d$ is hyperuniform iff

$$\hat{\eta}(B_r^\times) = o_{r\to 0}(R^d)$$

Hyperuniformity for tilings

Lemma

Let T be a nice inflation tiling with colored point process $\boldsymbol{\Lambda}$. If

$$\|\hat{\boldsymbol{\eta}}\|(B_r^{\times}) = o(r^d)$$

then Λ and Λ_i are hyperuniform for $i \in [\ell]$

Renormalization of diffraction ($\mathbb{R}^d$ version)

Theorem (R., Baake-Grimm-Gähler)

Let T be a nice inflation tiling with colored point process $\boldsymbol{\Lambda}$. Assume it has pure point spherical diffraction $\hat{\boldsymbol{\eta}}$. Let λ be the scaling constant and $\rho_2(A_{red})$ be the second largest eigenvalue of the inflation matrix. Then

$$\hat{\eta}(B_r^{\times}) = O(r^{d+\alpha-\epsilon})$$

where

$$\alpha := d - 2d\frac{\log|\rho_2(A_{red})|}{\log \lambda^d}$$

for $r \to 0$, all $\epsilon > 0$.

Hyperuniformity for inflation tilings: results

Name	$Var(\# T \cap B_R)$	$\hat{\eta}(B_r^\times)$ | |	
Fibonacci	$O(1)$	$O(r^4)$	BGG
Penrose	$O(R)$	$O(r^{8-\epsilon})$	FMV, R.
Octogonal	$O(R)$	$O(r^{8-\epsilon})$	FMV, R.
Shield	$O(R)$	$O(r^{4-\epsilon})$	R.
CAP	$O(R)$	$O(r^{4-\epsilon})$	R.
Hat	$O(R)$	$O(r^3)$	R.

- BGG: Baake, Grimm, Gähler
- FMV: Fuchs, Mosseri, Vidal

Some inflation tilings

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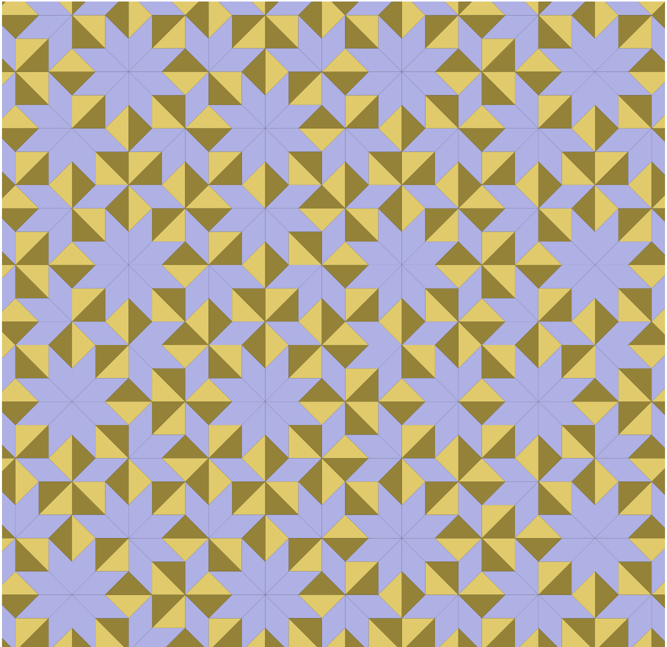

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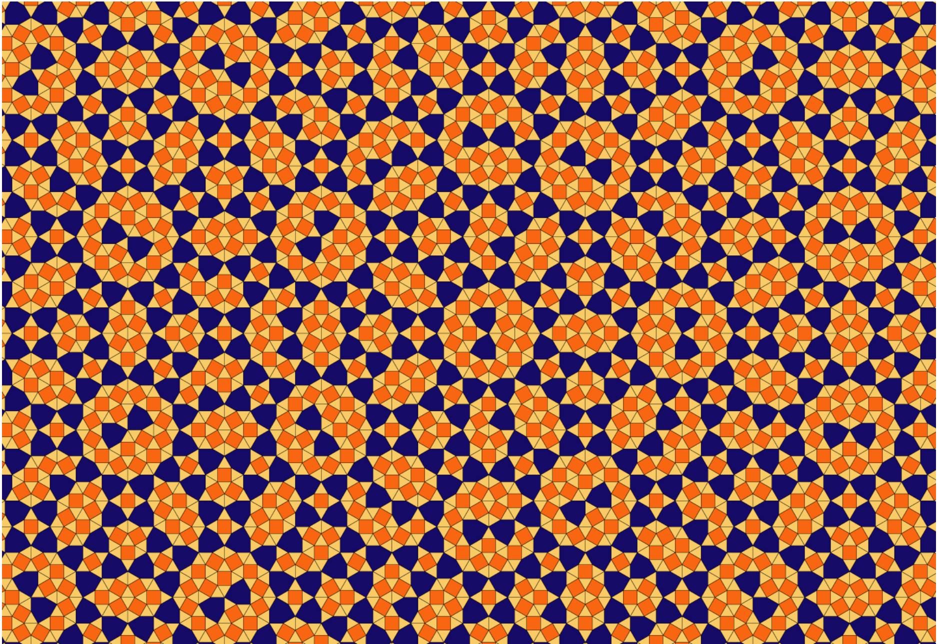

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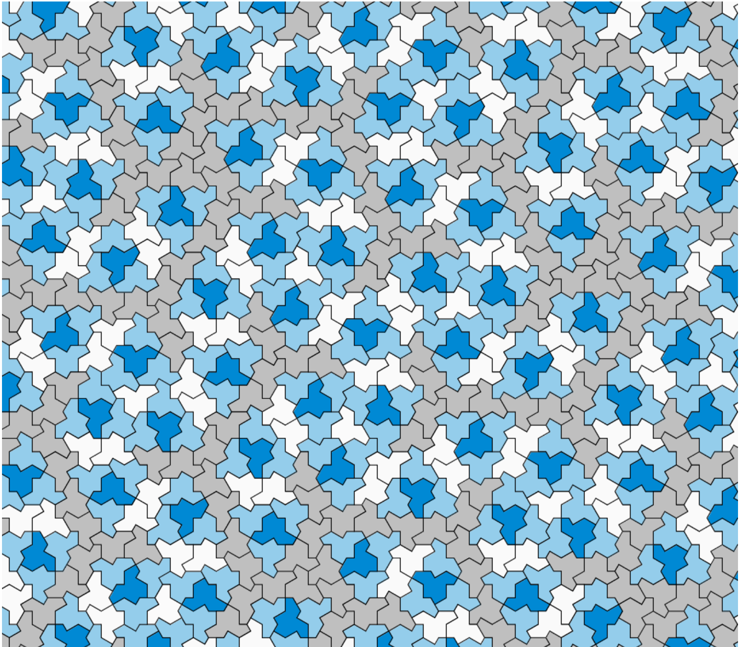

Thanks for listening!