NOTES FROM THE WORKSHOP ON APPROXIMATE GROUPS AND APERIODIC ORDER

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Based on joint work with Michael Björklund

1. Heat kernel hyperuniformity

Let $\Lambda$ be an invariant (locally square-summable) point process on hyperbolic space

$$\mathbb{H}^n = \text{SO}^+(1,n)/\text{SO}(n)$$

and let $\sigma_\Lambda$ denote the associated (centered) diffraction measure on $(0, +\infty) \cup i[0, \rho) \subset \mathbb{C}$, where $\rho = (n - 1)/2$. Recall from the talk that $\sigma_\Lambda$ is the unique positive Radon measure on $(0, +\infty) \cup i[0, \rho)$ satisfying

$$\text{Var} \left( \sum_{p \in \Lambda} f(p) \right) = \int_0^\infty |\hat{f}(\lambda)|^2 d\sigma_\Lambda^{(p)}(\lambda) + \int_0^\rho |\hat{f}(i\lambda)|^2 d\sigma_\Lambda^{(c)}(\lambda)$$

for all compactly supported continuous functions $f$ on $\mathbb{H}^n$, where $\sigma_\Lambda^{(c)}$, $\sigma_\Lambda^{(p)}$ are the restrictions of $\sigma_\Lambda$ to $(0, +\infty)$ and $i[0, \rho)$ respectively.

Assumption: We will assume throughout this note that $\sigma_\Lambda^{(c)} = 0$, saving us from any "super-Poissonian" density fluctuations of the process.

Goal: We want to show that such a point process $\Lambda$ is spectrally hyperuniform, i.e.

$$\sigma_\Lambda([0, \varepsilon]) = o(\varepsilon^3),$$

if and only if

$$\limsup_{t \to +\infty} t^{3/2} e^{2\rho^2 t} \text{Var} \left( \sum_{p \in \Lambda} h_t(p) \right) = 0,$$

where

$$h_t(x) = \int_0^\infty e^{-t(\rho^2 + \lambda^2)} \phi_\lambda(d(x, o)) \frac{d\lambda}{|c_\Lambda(\lambda)|^2}$$

is the heat kernel on $\mathbb{H}^n$. If $\Lambda$ satisfies equation \ref{eq:heat_kernel_hyperuni}, then we say that it is heat kernel hyperuniform.

1.1. Spectral hyperuniformity implies heat kernel hyperuniformity

Assume that $\Lambda$ is spectrally hyperuniform. We will first rewrite the variance of the periodized heat kernel into a suitable form and then formulate Lemma \ref{lem:heat_kernel_hyperuni}, where the heart of the proof lies.

Consider the function $f_t = e^{\rho^2 t} h_t$ on $\mathbb{H}^n$, so that

$$\hat{f}_t(\lambda) = e^{\rho^2 t} \hat{h}_t(\lambda) = e^{\rho^2 t} e^{-t(\rho^2 + \lambda^2)} = e^{-\rho^2 t} =: \psi(\sqrt{t}\lambda).$$

Using the diffraction measure $\sigma_\Lambda$, we rewrite the heat kernel variance as

$$t^{3/2} e^{2\rho^2 t} \text{Var} \left( \sum_{p \in \Lambda} h_t(p) \right) = t^{3/2} \text{Var} \left( \sum_{p \in \Lambda} f_t(p) \right) = t^{3/2} \int_0^\infty \psi(\sqrt{t}\lambda)^2 d\sigma_\Lambda(\lambda).$$

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Letting $s = \sqrt{t}$, we have to show that
$$s^3 \int_0^\infty \psi(s\lambda)^2 d\sigma_\Lambda(\lambda) \to 0, \quad s \to +\infty.$$ 

Set
$$\varphi(r) = -2\psi'(r)\psi(r) = 4r e^{-2r^2}$$
so that
$$\psi(s\lambda)^2 = s \int_\lambda^\infty \varphi(sr)dr.$$ 

By a standard change of variables,
$$\int_0^\infty \psi(s\lambda)^2 d\sigma_\Lambda(\lambda) = s \int_0^\infty \left( \int_\lambda^\infty \varphi(sr)dr \right) d\sigma_\Lambda(\lambda) = s \int_0^\infty \varphi(sr) \left( \int_0^1 d\sigma_\Lambda(\lambda) \right) dr$$
$$= s \int_0^\infty \varphi(sr) \sigma_\Lambda([0,r])dr.$$ 

This means that it is enough to show that
$$s^4 \int_0^\infty \varphi(sr) \sigma_\Lambda([0,r])dr \to 0, \quad s \to +\infty.$$ 

The core of the proof lies in the following Lemma.

**Lemma 1.1** (Björklund). Suppose that there are constants $\alpha, \beta > 0, \gamma \in (0, 1), A, B, C \geq 0$ and Borel functions $\omega, \varphi : [0, +\infty) \to [0, \infty)$ satisfying

(i) $\rho$ is increasing and such that

- for every $\varepsilon > 0$ there is an $r_\varepsilon \in (0, 1)$ with $r_\varepsilon \to 0$ such that $\omega(r) \leq A\varepsilon r^\alpha$ for all $r \in [0, r_\varepsilon^{1/\gamma}]$,
- there is a constant $M > 0$ such that $\omega(r) \leq Br^\beta$ for all $r \geq M$.

(ii) $\varphi$ is such that

1. $\int_0^\infty r^\alpha \varphi(r)dr < +\infty$,
2. $R^{\alpha/\gamma} \int_R^\infty \varphi(r)dr \to 0$ as $R \to +\infty$,
3. $R^{\alpha-\beta} \int_R^\infty r^\beta \varphi(r)dr \to 0$ as $R \to +\infty$.

Then
$$r_\varepsilon^{-(\alpha+1)} \int_0^\infty \varphi(r/r_\varepsilon)\omega(r)dr \to 0, \quad \varepsilon \to 0^+.$$ 

With this Lemma in mind, we set $\varphi(r) = 4r e^{-2r^2}$ and $\omega(r) = \sigma_\Lambda([0,r])$. By the assumed spectral hyperuniformity, there is a constant $A \geq 0$ and a sequence $r_\varepsilon \to 0$ such that
$$\omega(r) \leq A\varepsilon r^3, \quad r \in [0, r_\varepsilon^{1/2}],$$
so we can take $\alpha = 3, \gamma = 1/2$. Finding constants $B, \beta$ as in the Lemma above requires some more work.

**Lemma 1.2.** Let $\Lambda$ be an invariant point process on $\mathbb{H}^n$ with diffraction $\sigma_\Lambda$. Then $\sigma_\Lambda([0,R]) \ll R^n$ for sufficiently large $R > 0$.

**Remark 1.3.** When $\omega(r) = \sigma([0,r])$ is the diffraction measure of a point process, Lemma 1.1 is applicable for all rank one spaces. In particular, the crucial constants for Euclidean and hyperbolic space are


\[
\begin{array}{|c|c|c|}
\hline
G/K & \mathbb{R}^n & \mathbb{H}^n \\
\hline
\alpha & n & 3 \\
\beta & n & n \\
\hline
\end{array}
\]

Thus for hyperbolic space there is a \( B > 0 \) and an \( M > 0 \) such that \( \rho(r) \leq Br^n \) for all \( r \geq M \), so take \( \beta = n \).

Modulo the proof of Lemma 1.1 it suffices to verify points (1)-(3) for the function \( \varphi(r) = 4re^{-2r^2} \).

(1) we have
\[
4 \int_0^\infty r^4 e^{-2r^2} dr < +\infty.
\]

(2)
\[
R^6 \int_R^\infty 4re^{-2r^2} dr = 2R^6 e^{-2R^2} \longrightarrow 0, \quad R \rightarrow +\infty.
\]

(3)
\[
R^{3-n} \int_R^\infty r^n e^{-2r^2} dr = R^4 \int_1^\infty r^n e^{-2R^2 r^2} dr \longrightarrow 0, \quad R \rightarrow +\infty
\]

by dominated convergence.

Letting \( s_\varepsilon = r_\varepsilon^{-1} \), then Lemma 1.1 tells us that
\[
s_\varepsilon^4 \int_0^\infty \varphi(s_\varepsilon r)\sigma(\{0, r\}) dr \longrightarrow 0, \quad \varepsilon \rightarrow 0^+,
\]

which is what we wanted to show.

**Proof of Lemma 1.1.** Let \( I_\varepsilon = \int_0^\infty \varphi(r/r_\varepsilon)\rho(r) dr \). Then we use the assumed monotone growth and estimates on \( \omega \) to find that
\[
I_\varepsilon \leq \int_0^{r_\varepsilon^{-\gamma}} \varphi(r/r_\varepsilon)A\varepsilon r^\alpha dr + \omega(M) \int_1^{M^\gamma} \varphi(r/r_\varepsilon)dr + \int_M^\infty \varphi(r/r_\varepsilon)Br^\beta dr =: I_1 + I_2 + I_3.
\]

We need to show that \( I_j = o(\varepsilon^{-\alpha+1}) \) for \( j = 1, 2, 3 \) as \( \varepsilon \rightarrow 0^+ \).

\[
\begin{aligned}
I_1 & \leq A\varepsilon r_\varepsilon^{\alpha+1} \int_0^\infty \varphi(r)r^\alpha dr \\
I_2 & \leq \varphi(M)r_\varepsilon \int_{r_\varepsilon^{1-\gamma}}^{\infty} \varphi(r) dr \\
I_3 & \leq Br_\varepsilon^{\beta+1} \int_M^{\infty} \varphi(r)r^\beta dr.
\end{aligned}
\]

Dividing by \( r_\varepsilon^{\alpha+1} \), we get that
\[
\frac{I_\varepsilon}{r_\varepsilon^{\alpha+1}} = \frac{I_1 + I_2 + I_3}{r_\varepsilon^{\alpha+1}} \ll \varepsilon + \omega(M)(r_\varepsilon^{-\gamma})^{\alpha/\gamma} \int_{r_\varepsilon^{-\gamma}}^{\infty} \varphi(r) dr + r_\varepsilon^{-\alpha-beta} \int_M^{\infty} \varphi(r)r^\beta dr \rightarrow 0
\]

\[
\begin{aligned}
\longrightarrow 0 \\
\longrightarrow 0
\end{aligned}
\]

\[\blacksquare\]

1.2. **Heat kernel hyperuniformity implies spectral hyperuniformity**

Assume that equation (1.1) holds. We will show that \( \sigma(\{0, \varepsilon\}) = o(\varepsilon^3) \) as \( \varepsilon \rightarrow 0^+ \).

This is fortunately quite straightforward: Let \( \delta_\varepsilon > 0 \) and bound canonically from above,
\[
\text{Var}\left( \sum_{p \in \Lambda} h_t(p) \right) = \int_0^\infty \hat{h}_t(\lambda)^2 d\sigma(\lambda) \geq \int_0^{\delta_\varepsilon} \hat{h}_t(\lambda)^2 d\sigma(\lambda) = e^{-2\delta_\varepsilon^2 t} \int_0^{\delta_\varepsilon} e^{-2\lambda^2} d\sigma(\lambda).
\]
Assuming heat kernel hyperuniformity, there is for every \( \varepsilon > 0 \) a \( t_\varepsilon > 0 \) such that
\[
t^{3/2} \int_0^{\delta t} e^{-2t\lambda^2} d\sigma_\Lambda(\lambda) < \varepsilon, \quad \forall t \geq t_\varepsilon.
\]
In particular,
\[
t^{3/2} e^{-2t\delta_t^2} \sigma_\Lambda([0, \delta_t]) < \varepsilon, \quad \forall t \geq t_\varepsilon,
\]
so the question boils down to the following: If we can find \( \delta_t \to 0 \) as \( t \to +\infty \) such that
\[
\delta_t^{-3} \ll t^{3/2} e^{-2t\delta_t^2}, \quad \forall t \geq t_\varepsilon,
\]
then
\[
\frac{\sigma_\Lambda([0, \delta_t])}{\delta_t^3} \ll t^{3/2} e^{-2t\delta_t^2} \sigma_\Lambda([0, \delta_t]) < \varepsilon, \quad \forall t \geq t_\varepsilon,
\]
as desired.

Finding a solution \( \delta_t \) to equation (1.2) is equivalent to
\[
2t\delta_t^2 - 3\log(\delta_t) \leq \frac{3c}{2} \log(t)
\]
for some \( c > 0 \) for all sufficiently large \( t \). As an example, by setting \( c = 2 \) we get
\[
2t\delta_t^2 \leq 3\log(t\delta_t),
\]
for which \( \delta_t = t^{-1/2} \) is an example of a solution whenever \( t \geq e^{4/3} \).