On the number variance of invariant point processes

Mattias Byléhn
University of Gothenburg
bylehn@chalmers.se

23.08.2023

Joint work with Michael Björklund
Introduction

Let $G$ be a lcsc unimodular group with a pmp action on a probability space $(\Omega, \mathbb{P})$. An invariant simple point process on a homogeneous $G$-space $X = G/K$ is a $G$-equivariant map

$$\Lambda : \Omega \to \text{LF}(X), \quad \omega \mapsto \Lambda_\omega.$$  

Examples to have in mind are random lattice orbits and invariant Poisson processes.

We are interested in studying the number variance

$$\text{Var}(\#(\Lambda \cap B_R)) = \int_\Omega \left| \#(\Lambda_\omega \cap B_R) - i_\Lambda \text{Vol}_X(B_R) \right|^2 d\mathbb{P}(\omega), \quad R \gg 0.$$  

The number variance records "density fluctuations" of the random point set.
Let $G$ be a lcsc unimodular group with a pmp action on a probability space $(\Omega, \mathbb{P})$. An *invariant simple point process* on a homogeneous $G$-space $X = G/K$ is a $G$-equivariant map

$$\Lambda : \Omega \to \text{LF}(X), \quad \omega \mapsto \Lambda_\omega.$$ 

Examples to have in mind are random lattice orbits and invariant Poisson processes.

We are interested in studying the *number variance*

$$\text{Var}(\#(\Lambda \cap B_R)) = \int_{\Omega} \left| \#(\Lambda_\omega \cap B_R) - i_\Lambda \text{Vol}_X(B_R) \right|^2 d\mathbb{P}(\omega), \quad R \gg 0.$$ 

The number variance records ”density fluctuations” of the random point set.
Introduction

Let $G$ be a lcsc unimodular group with a pmp action on a probability space $(\Omega, \mathbb{P})$. An invariant simple point process on a homogeneous $G$-space $X = G/K$ is a $G$-equivariant map

$$\Lambda : \Omega \rightarrow \text{LF}(X), \quad \omega \mapsto \Lambda_\omega.$$  

Examples to have in mind are random lattice orbits and invariant Poisson processes. We are interested in studying the number variance

$$\text{Var}(\#(\Lambda \cap B_R)) = \int_{\Omega} \left| \#(\Lambda_\omega \cap B_R) - i_\Lambda \text{Vol}_X(B_R) \right|^2 d\mathbb{P}(\omega), \quad R \gg 0.$$  

The number variance records "density fluctuations" of the random point set.
Euclidean space: Beck’s Theorem

Consider $n$-dimensional Euclidean space $\mathbb{R}^n = (\text{SO}(n) \times \mathbb{R}^n)/\text{SO}(n)$ and $\Lambda$ an invariant point process on $\mathbb{R}^n$.

**Theorem (Beck, ’87)**

For sufficiently large $R_o > 0$ there is a constant $C = C(R_o, \Lambda) > 0$ such that

$$\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r)) dr \geq CR^{n-1} \quad \forall R \geq R_o.$$  

The number variance grows on average at least as fast as the measure of the boundary of the ball.
Euclidean diffraction

\( \text{Var}(\#(\Lambda \cap B_R)) \) is rotationally symmetric in \( \Lambda \), so converting hyperuniformity to the frequency domain will require the radial Fourier transform,

\[
\hat{f}(\lambda) = \int_{\mathbb{R}^d} f(x) \phi_\lambda(\|x\|) \, dx, \quad \lambda \geq 0, \; f \in C_c(\mathbb{R}^d) \text{ radial},
\]

where

\[
\phi_\lambda(t) = \text{const.} \, (\lambda t)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t) \\
= \text{const.} \, t^{2-n} \int_0^t (t^2 - s^2)^{\frac{n-3}{2}} \cos(\lambda s) \, ds
\]

are the bounded eigenfunctions of the radial Laplacian on \( \mathbb{R}^n \).

**Theorem (Bochner)**

Let \( \Lambda \) be an invariant point process on \( \mathbb{R}^n \). Then there is a unique positive Radon measure \( \sigma_\Lambda \) on \( (0, +\infty) \) such that

\[
\text{Var}\left( \sum_{p \in \Lambda} f(p) \right) = \int_0^\infty |\hat{f}(\lambda)|^2 \, d\sigma_\Lambda(\lambda).
\]

We refer to \( \sigma_\Lambda \) as the diffraction measure of \( \Lambda \).
Euclidean diffraction

Var(#(Λ ∩ B_R)) is rotationally symmetric in Λ, so converting hyperuniformity to the frequency domain will require the radial Fourier transform,

$$\hat{f}(\lambda) = \int_{\mathbb{R}^d} f(x) \phi_\lambda(||x||) \, dx, \quad \lambda \geq 0, \ f \in C_c(\mathbb{R}^d) \text{ radial},$$

where

$$\phi_\lambda(t) = \text{const.} (\lambda t)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t)$$

$$= \text{const.} t^{2-n} \int_0^t (t^2 - s^2)^{\frac{n-3}{2}} \cos(\lambda s) \, ds$$

are the bounded eigenfunctions of the radial Laplacian on \( \mathbb{R}^n \).

Theorem (Bochner)

Let \( \Lambda \) be an invariant point process on \( \mathbb{R}^n \). Then there is a unique positive Radon measure \( \sigma_\Lambda \) on \( (0, +\infty) \) such that

$$\text{Var} \left( \sum_{p \in \Lambda} f(p) \right) = \int_0^\infty |\hat{f}(\lambda)|^2 \, d\sigma_\Lambda(\lambda).$$

We refer to \( \sigma_\Lambda \) as the diffraction measure of \( \Lambda \).
Euclidean diffraction

Var(#(Λ ∩ B_R)) is rotationally symmetric in Λ, so converting hyperuniformity to the frequency domain will require the radial Fourier transform,

\[ \hat{f}(\lambda) = \int_{\mathbb{R}^d} f(x) \phi_\lambda(\|x\|) dx, \quad \lambda \geq 0, \; f \in C_c(\mathbb{R}^d) \text{ radial,} \]

where

\[ \phi_\lambda(t) = \text{const.} \; (\lambda t)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t) \]

\[ = \text{const.} \; t^{2-n} \int_0^t (t^2 - s^2)^{\frac{n-3}{2}} \cos(\lambda s) ds \]

are the bounded eigenfunctions of the radial Laplacian on \( \mathbb{R}^n \).

**Theorem (Bochner)**

*Let \( \Lambda \) be an invariant point process on \( \mathbb{R}^n \). Then there is a unique positive Radon measure \( \sigma_\Lambda \) on \( (0, +\infty) \) such that*

\[ \text{Var} \left( \sum_{p \in \Lambda} f(p) \right) = \int_0^\infty |\hat{f}(\lambda)|^2 d\sigma_\Lambda(\lambda). \]

*We refer to \( \sigma_\Lambda \) as the diffraction measure of \( \Lambda \).*
Examples

- **Periodic processes**: In the case of the integer lattice \( \mathbb{Z}^n \), Poisson summation yields

\[
\text{Var}\left( \sum_{p \in \Lambda_{\mathbb{Z}^n}} f(p) \right) = \sum_{\ell=1}^{\infty} r_n(\ell) |\hat{f}(\sqrt{\ell})|^2 ,
\]

where \( r_n(\ell) = \#\{\xi \in \mathbb{Z}^n : \|\xi\|^2 = \ell\} \), so \( \sigma_{\mathbb{Z}^n} = \sum_{\ell \geq 1} r_n(\ell) \delta_{\sqrt{\ell}} \).

- **Invariant Poisson process**: By the Plancherel formula,

\[
\text{Var}\left( \sum_{p \in \Lambda_{\text{Poi}}} f(p) \right) = \int_{\mathbb{R}^n} |f(x)|^2 dx = \int_0^{\infty} |\hat{f}(\lambda)|^2 \lambda^{n-1} d\lambda ,
\]

so \( d\sigma_{\text{Poi}}(\lambda) = \lambda^{d-1} d\lambda \), the Plancherel measure.
Examples

- **Periodic processes**: In the case of the integer lattice $\mathbb{Z}^n$, Poisson summation yields

$$\text{Var}\left( \sum_{p \in \Lambda_{\mathbb{Z}^n}} f(p) \right) = \sum_{\ell=1}^{\infty} r_n(\ell) |\hat{f}(\sqrt{\ell})|^2,$$

where $r_n(\ell) = \#\{\xi \in \mathbb{Z}^n : \|\xi\|^2 = \ell\}$, so $\sigma_{\mathbb{Z}^n} = \sum_{\ell \geq 1} r_n(\ell) \delta_{\sqrt{\ell}}$.

- **Invariant Poisson process**: By the Plancherel formula,

$$\text{Var}\left( \sum_{p \in \Lambda_{\text{Poi}}} f(p) \right) = \int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{0}^{\infty} |\hat{f}(\lambda)|^2 \lambda^{n-1} \, d\lambda,$$

so $d\sigma_{\text{Poi}}(\lambda) = \lambda^{d-1} \, d\lambda$, the Plancherel measure.
We have
\[
\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r)) \, dr = \int_0^\infty \left( \frac{1}{R} \int_0^R |\widehat{\chi}_{B_r}(\lambda)|^2 \, dr \right) d\sigma_\Lambda(\lambda).
\]

Fixing $\lambda > 0$, one computes $\widehat{\chi}_{B_r}(\lambda) = (r/\lambda)^{n/2} J_{n/2}(\lambda r)$ and show that
\[
\frac{1}{R\lambda^n} \int_0^R r^n J_{n/2}(\lambda r)^2 \, dr \asymp n \frac{R^{n-1}}{\lambda^{n+1}}, \quad R \gg 0.
\]

This asymptotic is uniform in $\lambda \in [\lambda_o, +\infty)$ and $R \geq R_o$ for $\lambda_o > R_o^{-1}$, so we can choose the lower bound
\[
\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r)) \, dr \gg_n \left( \int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}} \right) R^{n-1}.
\]
Idea of proof for Beck’s Theorem

We have
\[
\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r)) dr = \int_0^\infty \left( \frac{1}{R} \int_0^R |\widehat{\chi}_{B_r}(\lambda)|^2 dr \right) d\sigma_\Lambda(\lambda) .
\]

Fixing \( \lambda > 0 \), one computes \( \widehat{\chi}_{B_r}(\lambda) = (r/\lambda)^{n/2} J_{n/2}(\lambda r) \) and show that
\[
\frac{1}{R\lambda^n} \int_0^R r^n J_{n/2}(\lambda r)^2 dr \asymp_n \frac{R^{n-1}}{\lambda^{n+1}} , \quad R \gg 0 .
\]

This asymptotic is uniform in \( \lambda \in [\lambda_o, +\infty) \) and \( R \geq R_o \) for \( \lambda_o > R_o^{-1} \), so we can choose the lower bound
\[
\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r)) dr \gg_n \left( \int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}} \right) R^{n-1} .
\]
Idea of proof for Beck’s Theorem

We have

\[
\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r))dr = \int_0^\infty \left( \frac{1}{R} \int_0^R |\hat{\chi}_{B_r}(\lambda)|^2 dr \right) d\sigma_\Lambda(\lambda). 
\]

Fixing \( \lambda > 0 \), one computes \( \hat{\chi}_{B_r}(\lambda) = (r/\lambda)^{n/2} J_{n/2}(\lambda r) \) and show that

\[
\frac{1}{R\lambda^n} \int_0^R r^n J_{n/2}(\lambda r)^2 dr \asymp_n \frac{R^{n-1}}{\lambda^{n+1}}, \quad R \gg 0. 
\]

This asymptotic is uniform in \( \lambda \in [\lambda_o, +\infty) \) and \( R \geq R_o \) for \( \lambda_o > R_o^{-1} \), so we can choose the lower bound

\[
\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r))dr \gg_n \left( \int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}} \right) R^{n-1}. 
\]
There are two things to note:

1. If there is a $\lambda_o > 0$ such that $\sigma_\Lambda([0, \lambda_o]) = 0$, then

   $$\frac{1}{R} \int_0^R \text{Var}(\#(\Lambda \cap B_r))dr \asymp_n \left( \int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}} \right) R^{n-1}.$$ 

   We call such point processes $\Lambda$ *stealthy*. (For example $\Lambda_{\mathbb{Z}^n}$.)

2. The number variance might behave differently along specific subsequences. We show that there is a sequence $R_j \to +\infty$ such that

   $$\text{Var}(\#(\Lambda_{\mathbb{Z}^5} \cap B_{R_j})) = o(R_j^4).$$
Hyperuniformity

From the proof sketch of Beck’s Theorem, we are left with investigating integrals of the form

\[ \int_{0}^{\lambda_0} \frac{d\sigma_{\Lambda}(\lambda)}{\lambda^{n+1}}, \quad \lambda_0 \approx 0. \]

**Definition**

An invariant point process \( \Lambda \) is *spectrally hyperuniform* if

\[ \limsup_{\varepsilon \to 0^+} \frac{\sigma_{\Lambda}([0, \varepsilon])}{\varepsilon^n} = 0. \]

**Theorem (Björklund-Hartnick, ’22)**

Let \( \Lambda \) be an invariant point process in \( \mathbb{R}^d \). Then

\[ \limsup_{R \to +\infty} \frac{\text{Var}(\#(\Lambda \cap B_R))}{R^n} = 0 \iff \limsup_{\varepsilon \to 0^+} \frac{\sigma_{\Lambda}([0, \varepsilon])}{\varepsilon^n} = 0. \]

Examples include lattices, arithmetic model sets, certain determinantal point processes and more.
Hyperuniformity

From the proof sketch of Beck’s Theorem, we are left with investigating integrals of the form

\[ \int_0^{\lambda_o} d\sigma_\Lambda(\lambda) \frac{\lambda}{\lambda^{n+1}}, \quad \lambda_o \approx 0. \]

**Definition**

An invariant point process \( \Lambda \) is *spectrally hyperuniform* if

\[ \limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda([0, \varepsilon])}{\varepsilon^n} = 0. \]

**Theorem (Björklund-Hartnick, ’22)**

*Let \( \Lambda \) be an invariant point process in \( \mathbb{R}^d \). Then*

\[ \limsup_{R \to +\infty} \frac{\text{Var}(\#(\Lambda \cap B_R))}{R^n} = 0 \iff \limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda([0, \varepsilon])}{\varepsilon^n} = 0. \]

Examples include lattices, arithmetic model sets, certain determinantal point processes and more.
What about other geometries?
Hyperbolic space

Consider the one-sheeted hyperboloid

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : [x, x] = x_0^2 - x_1^2 - \ldots - x_n^2 = 1, \ x_0 > 0 \right\} \cong \text{SO}^+(1, n)/\text{SO}(n).$$

with the metric $d(x, y) = \text{arccosh}([x, y])$.

The $\text{SO}^+(1, n)$-invariant reference measure is in polar/Cartan coordinates given by

$$dm_n(\cosh(t), \sinh(t)v) = \sinh(t)^{n-1}dt\,d\varsigma_{n-1}(v),$$

and the volume growth of the centered $\text{SO}(n)$-invariant ball $B_R = B_R(o)$ is

$$\text{Vol}_{\mathbb{H}^n}(B_R) = \varsigma_{n-1}(S^{n-1}) \int_0^R \sinh(t)^{n-1}dt \asymp e^{(n-1)R}.$$
Hyperbolic space

Consider the one-sheeted hyperboloid

\[ \mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : [x, x] = x_0^2 - x_1^2 - \ldots - x_n^2 = 1, \ x_0 > 0 \right\} \cong \text{SO}^+(1, n)/\text{SO}(n). \]

with the metric \( d(x, y) = \text{arccosh}([x, y]). \)

The \( \text{SO}^+(1, n) \)-invariant reference measure is in polar/Cartan coordinates given by

\[ dm_n(\cosh(t), \sinh(t)v) = \sinh(t)^{n-1} dt d\varsigma_{n-1}(v), \]

and the volume growth of the centered \( \text{SO}(n) \)-invariant ball \( B_R = B_R(o) \) is

\[ \text{Vol}_{\mathbb{H}^n}(B_R) = \varsigma_{n-1}(S^{n-1}) \int_0^R \sinh(t)^{n-1} dt \asymp e^{(n-1)R}. \]
Hyperbolic space

Consider the one-sheeted hyperboloid

\[ H^n = \left\{ x \in \mathbb{R}^{n+1} : [x, x] = x_0^2 - x_1^2 - \ldots - x_n^2 = 1, \ x_0 > 0 \right\} \cong \text{SO}^+(1, n)/\text{SO}(n). \]

with the metric \( d(x, y) = \text{arccosh}([x, y]). \)

The \( \text{SO}^+(1, n) \)-invariant reference measure is in polar/Cartan coordinates given by

\[ dm_n(\cosh(t), \sinh(t)v) = \sinh(t)^{n-1} dt\ d\varsigma_{n-1}(v), \]

and the volume growth of the centered \( \text{SO}(n) \)-invariant ball \( B_R = B_R(o) \) is

\[ \text{Vol}_{H^n}(B_R) = \varsigma_{n-1}(S^{n-1}) \int_0^R \sinh(t)^{n-1} dt \asymp e^{(n-1)R}. \]
Fourier analysis on $\mathbb{H}^d$

We again consider the radial Fourier transform, which for $\mathbb{H}^n$ is

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(x) \phi_\lambda(\ell(x)) dm_n(x), \quad f \in C_c(\mathbb{H}^n) \text{ radial},$$

where $\ell(x) = d(x, o)$ and

$$\phi_\lambda(\ell(x)) = \text{const.} \sinh(\ell(x))^{2-n} \int_0^{\ell(x)} (\cosh(\ell(x)) - \cosh(s))^\frac{n-3}{2} \cos(\lambda s) ds$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{H}^n$.

Here we allow for $\lambda \in [0, +\infty) \cup i[0, (n - 1)/2]$.

There is moreover a Plancherel formula

$$\int_{\mathbb{H}^n} |f(x)|^2 dm_n(x) = \int_0^\infty |\hat{f}(\lambda)|^2 \frac{d\lambda}{|c_n(\lambda)|^2}$$

where $c_n(\lambda) \propto \Gamma(i\lambda)/\Gamma(i\lambda + (n - 1)/2)$ is the Harish-Chandra c-function.
We again consider the radial Fourier transform, which for $\mathbb{H}^n$ is

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(x)\phi_\lambda(\ell(x)) \, dm_n(x), \quad f \in C_c(\mathbb{H}^n) \text{ radial},$$

where $\ell(x) = d(x, o)$ and

$$\phi_\lambda(\ell(x)) = \text{const. } \sinh(\ell(x))^{2-n} \int_0^{\ell(x)} (\cosh(\ell(x)) - \cosh(s))^{\frac{n-3}{2}} \cos(\lambda s) ds$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{H}^n$.

Here we allow for $\lambda \in [0, +\infty) \cup i[0, (n-1)/2]$. There is moreover a Plancherel formula

$$\int_{\mathbb{H}^n} |f(x)|^2 \, dm_n(x) = \int_0^\infty |\hat{f}(\lambda)|^2 \frac{d\lambda}{|c_n(\lambda)|^2}$$

where $c_n(\lambda) \propto \Gamma(i\lambda)/\Gamma(i\lambda + (n-1)/2)$ is the Harish-Chandra $c$-function.
Fourier analysis on $\mathbb{H}^d$

We again consider the radial Fourier transform, which for $\mathbb{H}^n$ is

$$\hat{f}(\lambda) = \int_{\mathbb{H}^n} f(x) \phi_\lambda(\ell(x)) dm_n(x), \quad f \in C_c(\mathbb{H}^n) \text{ radial},$$

where $\ell(x) = d(x, o)$ and

$$\phi_\lambda(\ell(x)) = \text{const.} \sinh(\ell(x))^{2-n} \int_0^\ell(x) (\cosh(\ell(x)) - \cosh(s))^{\frac{n-3}{2}} \cos(\lambda s) ds$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{H}^n$.

Here we allow for $\lambda \in [0, +\infty) \cup i[0, (n - 1)/2]$.

There is moreover a Plancherel formula

$$\int_{\mathbb{H}^n} |f(x)|^2 dm_n(x) = \int_0^\infty |\hat{f}(\lambda)|^2 \frac{d\lambda}{|c_n(\lambda)|^2}$$

where $c_n(\lambda) \propto \Gamma(i\lambda)/\Gamma(i\lambda + (n - 1)/2)$ is the Harish-Chandra c-function.
Hyperbolic diffraction

The existence of a diffraction measure for invariant point processes is slightly more subtle for $\mathbb{H}^n$.

Theorem (Krein, Gelfand-Vilenkin)

Let $\Lambda$ be an invariant point process on $\mathbb{H}^n$. Then there is a unique positive Radon measure $\sigma^{(p)}_\Lambda$ on $(0, +\infty)$ and a unique positive finite measure $\sigma^{(c)}_\Lambda$ on $[0, (n-1)/2)$ such that

$$\operatorname{Var}\left( \sum_{p \in \Lambda} f(p) \right) = \int_0^\infty |\hat{f}(\lambda)|^2 \, d\sigma^{(p)}_\Lambda(\lambda) + \int_0^{\frac{n-1}{2}} |\hat{f}(i\lambda)|^2 \, d\sigma^{(c)}_\Lambda(\lambda).$$

We refer to $\sigma^{(p)}_\Lambda$, $\sigma^{(c)}_\Lambda$ as the principal and complementary diffraction measures of $\Lambda$ respectively.

Example (Hyperbolic Poisson diffraction)

The diffraction of the $\operatorname{Vol}_{d}$-Poisson point process is $d\sigma^{(p)}_{\text{Poi}}(\lambda) = |c_n(\lambda)|^{-2} \, d\lambda$ and $d\sigma^{(c)}_\Lambda(\lambda) = 0$ by the Plancherel formula.
Hyperbolic diffraction

The existence of a diffraction measure for invariant point processes is slightly more subtle for $\mathbb{H}^n$.

**Theorem (Krein, Gelfand-Vilenkin)**

Let $\Lambda$ be an invariant point process on $\mathbb{H}^n$. Then there is a unique positive Radon measure $\sigma^{(p)}_\Lambda$ on $(0, +\infty)$ and a unique positive finite measure $\sigma^{(c)}_\Lambda$ on $[0, (n-1)/2)$ such that

$$\text{Var} \left( \sum_{p \in \Lambda} f(p) \right) = \int_0^\infty |\hat{f}(\lambda)|^2 d\sigma^{(p)}_\Lambda(\lambda) + \int_0^{n-1 \over 2} |\hat{f}(i\lambda)|^2 d\sigma^{(c)}_\Lambda(\lambda).$$

We refer to $\sigma^{(p)}_\Lambda, \sigma^{(c)}_\Lambda$ as the principal and complementary diffraction measures of $\Lambda$ respectively.

**Example (Hyperbolic Poisson diffraction)**

The diffraction of the Vol$_d$-Poisson point process is $d\sigma^{(p)}_{\text{Poi}}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma^{(c)}_\Lambda(\lambda) = 0$ by the Plancherel formula.
The existence of a diffraction measure for invariant point processes is slightly more subtle for $\mathbb{H}^n$.

**Theorem (Krein, Gelfand-Vilenkin)**

Let $\Lambda$ be an invariant point process on $\mathbb{H}^n$. Then there is a unique positive Radon measure $\sigma^{(p)}_\Lambda$ on $(0, +\infty)$ and a unique positive finite measure $\sigma^{(c)}_\Lambda$ on $[0, (n - 1)/2)$ such that

$$\text{Var}\left(\sum_{p \in \Lambda} f(p)\right) = \int_0^\infty |\hat{f}(\lambda)|^2 d\sigma^{(p)}_\Lambda(\lambda) + \int_0^{\frac{n-1}{2}} |\hat{f}(i\lambda)|^2 d\sigma^{(c)}_\Lambda(\lambda).$$

We refer to $\sigma^{(p)}_\Lambda, \sigma^{(c)}_\Lambda$ as the principal and complementary diffraction measures of $\Lambda$ respectively.

**Example (Hyperbolic Poisson diffraction)**

The diffraction of the $\text{Vol}_d$-Poisson point process is $d\sigma^{(p)}_{\text{Poi}}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma^{(c)}_\Lambda(\lambda) = 0$ by the Plancherel formula.
Hyperbolic (non-)hyperuniformity

An analogue of Beck’s Theorem

**Theorem (Björklund-B.)**

Let $\Lambda$ be an invariant point process on $\mathbb{H}^d$.

1. If $\sigma^{(c)}_\Lambda \neq 0$ then

$$
\lim_{R \to +\infty} \frac{\text{Var}(\#(\Lambda \cap B_R))}{e^{(n-1)R}} \xrightarrow{R \to +\infty} +\infty
$$

2. If $\sigma^{(c)}_\Lambda = 0$ then

$$
\lim_{R \to +\infty} \frac{1}{R} \int_0^R \frac{\text{Var}(\#(\Lambda \cap B_r))}{e^{(n-1)r}} dr \gg n \int_{\lambda_0}^{\infty} |c_{n+2}(\lambda)|^2 d\sigma^{(p)}_\Lambda (\lambda) > 0.
$$

In particular, $\Lambda$ is not ”geometrically hyperuniform”.

Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0, \varepsilon])$ with $\sigma_{\text{Poi}}([0, \varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma^{(p)}_{\text{Poi}}(\lambda) = |c_n(\lambda)|^{-2}d\lambda$ and $d\sigma^{(c)}_{\text{Poi}} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma^{(p)}_{\text{Poi}}([0, \varepsilon]) \approx \varepsilon^3.$$

Definition

An invariant point process $\Lambda$ on $\mathbb{H}^n$ is *spectrally hyperuniform* if $\sigma^{(c)}_\Lambda = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma^{(p)}_\Lambda([0, \varepsilon])}{\varepsilon^3} = 0.$$ 

Geometrically, we compared the number variance with

$$\text{Vol}_{\mathbb{H}^n}(B_R) = \hat{\chi}_{B_R}(i\frac{n-1}{2}),$$

but $\lambda = i\frac{n-1}{2}$ does **not** lie in the support of $\sigma^{(p)}_{\text{Poi}}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$. 
Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0, \varepsilon])$ with $\sigma_{\text{Poi}}([0, \varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma_{\text{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2}d\lambda$ and $d\sigma_{\text{Poi}}^{(c)} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma_{\text{Poi}}^{(p)}([0, \varepsilon]) \approx \varepsilon^3.$$ 

Definition

An invariant point process $\Lambda$ on $\mathbb{H}^n$ is \textit{spectrally hyperuniform} if $\sigma_{\Lambda}^{(c)} = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_{\Lambda}^{(p)}([0, \varepsilon])}{\varepsilon^3} = 0.$$ 

Geometrically, we compared the number variance with

$$\text{Vol}_{\mathbb{H}^n}(B_R) = \hat{\chi}_{B_R}(i^{\frac{n-1}{2}}),$$

but $\lambda = i^{\frac{n-1}{2}}$ does \textbf{not} lie in the support of $\sigma_{\text{Poi}}^{(p)}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$. 

29 / 40
Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0, \varepsilon])$ with $\sigma_{\text{Poi}}([0, \varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma_{\text{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2}d\lambda$ and $d\sigma_{\text{Poi}}^{(c)} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma_{\text{Poi}}^{(p)}([0, \varepsilon]) \approx \varepsilon^3.$$

Definition

An invariant point process $\Lambda$ on $\mathbb{H}^n$ is **spectrally hyperuniform** if $\sigma_\Lambda^{(c)} = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda^{(p)}([0, \varepsilon])}{\varepsilon^3} = 0.$$

Geometrically, we compared the number variance with

$$\text{Vol}_{\mathbb{H}^n}(B_R) = \tilde{\chi}_{B_R}(i \frac{n-1}{2}),$$

but $\lambda = i \frac{n-1}{2}$ does **not** lie in the support of $\sigma_{\text{Poi}}^{(p)}$. Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$.  

Geometric hyperuniformity
Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0, \varepsilon])$ with $\sigma_{\text{Poi}}([0, \varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma^{(p)}_{\text{Poi}}(\lambda) = |c_n(\lambda)|^{-2}d\lambda$ and $d\sigma^{(c)}_{\text{Poi}} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma^{(p)}_{\text{Poi}}([0, \varepsilon]) \approx \varepsilon^3.$$ 

**Definition**

An invariant point process $\Lambda$ on $\mathbb{H}^n$ is *spectrally hyperuniform* if $\sigma^{(c)}_\Lambda = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma^{(p)}_\Lambda([0, \varepsilon])}{\varepsilon^3} = 0.$$ 

Geometrically, we compared the number variance with

$$\text{Vol}_{\mathbb{H}^n}(B_R) = \widehat{\chi}_{\mathbb{H}^n}(i \frac{n-1}{2}),$$

but $\lambda = i \frac{n-1}{2}$ does **not** lie in the support of $\sigma^{(p)}_{\text{Poi}}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$. 

31 / 40
What is the "geometric" counterpart of spectral hyperuniformity?
It is in general quite hard to compute diffraction measures of point processes. We consider the heat kernel

$$h_t(x) = \int_0^\infty e^{-t((n-1)/2)^2+\lambda^2)} \phi_\lambda(\ell(x)) \frac{d\lambda}{|c_n(\lambda)|^2}, \quad t > 0.$$ 

**Theorem (Björklund-B., in the works)**

An invariant point process $\Lambda$ on $\mathbb{H}^n$ is spectrally hyperuniform if and only if

$$\limsup_{t \to +\infty} t^{3/2} e^{(n-1)^2 t/2} \text{Var} \left( \sum_{p \in \Lambda} h_t(p) \right) = 0.$$
Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?

- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?

- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?

- Is there a similar notion of ”order”/”uniformity” for processes admitting complementary spectrum?

- What point processes are stealthy?
Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?

- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?

- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?

- Is there a similar notion of "order"/"uniformity" for processes admitting complementary spectrum?

- What point processes are stealthy?
What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?

Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?

Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?

Is there a similar notion of ”order”/”uniformity” for processes admitting complementary spectrum?

What point processes are stealthy?
Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?

- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?

- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?

- Is there a similar notion of ”order”/”uniformity” for processes admitting complementary spectrum?

- What point processes are stealthy?
Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?

- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?

- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?

- Is there a similar notion of ”order”/”uniformity” for processes admitting complementary spectrum?

- What point processes are stealthy?
Thank you!
Sources

1. J. Beck, ”Irregularities of distribution”, ACTA matematica vol. 159, 1987:


2. M. Björklund, T. Hartnick, ”Hyperuniformity and non-hyperuniformity of quasicrystals”, 2022:


   https://gdz.sub.uni-goettingen.de/id/PPN358147735_0059?tify=%7B%22pages%22%3A%5B197%5D%2C%22pan%22%3A%7B%22x%22%3A0.847%2C%22y%22%3A0.788%7D%2C%22view%22%3A%22info%22%2C%22zoom%22%3A0.352%7D