On the number variance of invariant point processes

Mattias Byléhn
University of Gothenburg
bylehn@chalmers.se

23.08.2023

Joint work with Michael Björklund

Introduction

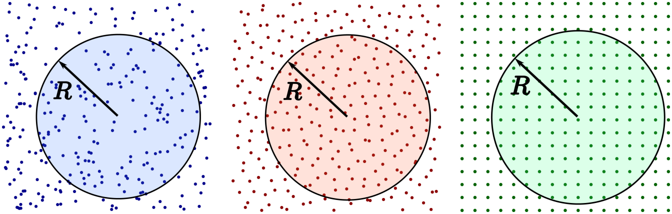
Let G be a lcsc unimodular group with a pmp action on a probability space $(\Omega, \mathbb{P})$. An *invariant simple point process* on a homogeneous G-space $X = G/K$ is a G-equivariant map

$$\Lambda : \Omega \to \mathrm{LF}(X)\,, \quad \omega \mapsto \Lambda_\omega\,.$$

Examples to have in mind are random lattice orbits and invariant Poisson processes.

We are interested in studying the *number variance*

$$\mathrm{Var}(\#(\Lambda \cap B_R)) = \int_\Omega \left| \#(\Lambda_\omega \cap B_R) - i_\Lambda \mathrm{Vol}_X(B_R) \right|^2 d\mathbb{P}(\omega)\,, \quad R \gg 0\,.$$



The number variance records "density fluctuations" of the random point set.

Introduction

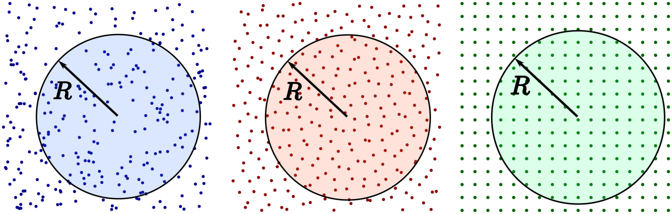
Let G be a lcsc unimodular group with a pmp action on a probability space $(\Omega, \mathbb{P})$. An *invariant simple point process* on a homogeneous G-space $X = G/K$ is a G-equivariant map

$$\Lambda : \Omega \to \mathrm{LF}(X)\,, \quad \omega \mapsto \Lambda_\omega\,.$$

Examples to have in mind are random lattice orbits and invariant Poisson processes.

We are interested in studying the *number variance*

$$\mathrm{Var}(\#(\Lambda \cap B_R)) = \int_\Omega \left| \#(\Lambda_\omega \cap B_R) - i_\Lambda \mathrm{Vol}_X(B_R) \right|^2 d\mathbb{P}(\omega)\,, \quad R \gg 0\,.$$



The number variance records "density fluctuations" of the random point set.

Introduction

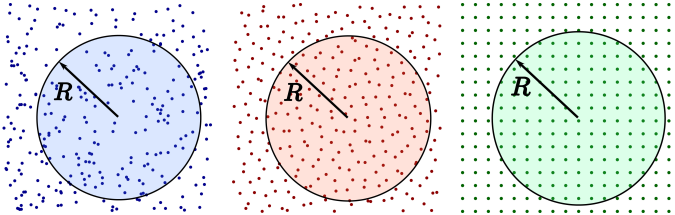
Let G be a lcsc unimodular group with a pmp action on a probability space $(\Omega, \mathbb{P})$. An *invariant simple point process* on a homogeneous G-space $X = G/K$ is a G-equivariant map

$$\Lambda : \Omega \to \mathrm{LF}(X)\,, \quad \omega \mapsto \Lambda_\omega\,.$$

Examples to have in mind are random lattice orbits and invariant Poisson processes.

We are interested in studying the *number variance*

$$\mathrm{Var}(\#(\Lambda \cap B_R)) = \int_\Omega \left| \#(\Lambda_\omega \cap B_R) - i_\Lambda \mathrm{Vol}_X(B_R) \right|^2 d\mathbb{P}(\omega)\,, \quad R \gg 0\,.$$



The number variance records ”density fluctuations” of the random point set.

Euclidean space: Beck's Theorem

Consider n-dimensional Euclidean space $\mathbb{R}^n = (\mathrm{SO}(n) \ltimes \mathbb{R}^n)/\mathrm{SO}(n)$ and Λ an invariant point process on $\mathbb{R}^n$.

Theorem (Beck, '87)

For su ciently large $R_o > 0$ there is a constant $C = C(R_o, \Lambda) > 0$ such that

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda \cap B_r))dr \geq CR^{n-1} \quad \forall R \geq R_o\,.$$

The number variance grows on average at least as fast as the measure of the boundary of the ball.

Euclidean diffraction

$\mathrm{Var}(\#(\Lambda \cap B_R))$ is rotationally symmetric in Λ, so converting hyperuniformity to the frequency domain will require the radial Fourier transform,

$$\widehat{f}(\lambda) = \int_{\mathsf{R}^d} f(x)\phi_\lambda(\|x\|)dx\,, \quad \lambda \geq 0,\ f \in C_c(\mathbb{R}^d) \text{ radial,}$$

where

$$\begin{aligned}\phi_\lambda(t) &= \text{const.}\,(\lambda t)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t)\\ &= \text{const.}\,t^{2-n}\int_0^t (t^2 - s^2)^{\frac{n-3}{2}}\cos(\lambda s)ds\end{aligned}$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{R}^n$.

Theorem (Bochner)

Let Λ be an invariant point process on $\mathbb{R}^n$. Then there is a unique positive Radon measure σ_Λ on $(0, +\infty)$ such that

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda(\lambda)\,.$$

We refer to σ_Λ as the *diffraction measure* of Λ.

Euclidean diffraction

$\mathrm{Var}(\#(\Lambda \cap B_R))$ is rotationally symmetric in Λ, so converting hyperuniformity to the frequency domain will require the radial Fourier transform,

$$\widehat{f}(\lambda) = \int_{\mathsf{R}^d} f(x)\phi_\lambda(\|x\|)dx\,, \quad \lambda \geq 0,\ f \in C_c(\mathbb{R}^d) \text{ radial,}$$

where

$$\begin{aligned}\phi_\lambda(t) &= \text{const.}\,(\lambda t)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t)\\ &= \text{const.}\, t^{2-n}\int_0^t (t^2 - s^2)^{\frac{n-3}{2}}\cos(\lambda s)ds\end{aligned}$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{R}^n$.

Theorem (Bochner)

Let Λ be an invariant point process on $\mathbb{R}^n$. Then there is a unique positive Radon measure σ_Λ on $(0, +\infty)$ such that

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda(\lambda)\,.$$

We refer to σ_Λ as the *diffraction measure* of Λ.

Euclidean diffraction

$\mathrm{Var}(\#(\Lambda \cap B_R))$ is rotationally symmetric in Λ, so converting hyperuniformity to the frequency domain will require the radial Fourier transform,

$$\widehat{f}(\lambda) = \int_{\mathsf{R}^d} f(x)\phi_\lambda(\|x\|)dx\,, \quad \lambda \geq 0,\ f \in C_c(\mathbb{R}^d) \text{ radial},$$

where

$$\begin{aligned}\phi_\lambda(t) &= \text{const.}\,(\lambda t)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda t) \\ &= \text{const.}\, t^{2-n} \int_0^t (t^2 - s^2)^{\frac{n-3}{2}} \cos(\lambda s) ds\end{aligned}$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{R}^n$.

Theorem (Bochner)

Let Λ be an invariant point process on $\mathbb{R}^n$. Then there is a unique positive Radon measure σ_Λ on $(0, +\infty)$ such that

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda(\lambda)\,.$$

We refer to σ_Λ as the *di raction measure* of Λ.

Examples

- Periodic processes: In the case of the integer lattice $\mathbb{Z}^n$, Poisson summation yields

$$\mathrm{Var}\Big(\sum_{p\in\Lambda_{\mathbb{Z}^n}} f(p)\Big) = \sum_{\ell=1}^{\infty} r_n(\ell)|\widehat{f}(\sqrt{\ell})|^2\,,$$

where $r_n(\ell) = \#\{\xi \in \mathbb{Z}^n : \|\xi\|^2 = \ell\}$, so $\sigma_{\mathsf{Z}^n} = \sum_{\ell\geq 1} r_n(\ell)\delta_{\sqrt{\ell}}$.

- Invariant Poisson process: By the Plancherel formula,

$$\mathrm{Var}\Big(\sum_{p\in\Lambda_{\mathrm{Poi}}} f(p)\Big) = \int_{\mathsf{R}^n} |f(x)|^2 dx = \int_0^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{n-1} d\lambda\,,$$

so $d\sigma_{\mathrm{Poi}}(\lambda) = \lambda^{d-1} d\lambda$, the Plancherel measure.

Examples

- Periodic processes: In the case of the integer lattice $\mathbb{Z}^n$, Poisson summation yields

$$\mathrm{Var}\Big(\sum_{p\in\Lambda_{\mathbb{Z}^n}} f(p)\Big) = \sum_{\ell=1}^{\infty} r_n(\ell)|\widehat{f}(\sqrt{\ell})|^2\,,$$

where $r_n(\ell) = \#\{\xi \in \mathbb{Z}^n : \|\xi\|^2 = \ell\}$, so $\sigma_{\mathsf{Z}^n} = \sum_{\ell\geq 1} r_n(\ell)\delta_{\sqrt{\ell}}$.

- Invariant Poisson process: By the Plancherel formula,

$$\mathrm{Var}\Big(\sum_{p\in\Lambda_{\mathrm{Poi}}} f(p)\Big) = \int_{\mathsf{R}^n} |f(x)|^2 dx = \int_0^{\infty} |\widehat{f}(\lambda)|^2 \lambda^{n-1} d\lambda\,,$$

so $d\sigma_{\mathrm{Poi}}(\lambda) = \lambda^{d-1} d\lambda$, the Plancherel measure.

Idea of proof for Beck's Theorem

We have

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda \cap B_r))dr = \int_0^\infty \Big(\frac{1}{R}\int_0^R |\widehat{\chi}_{B_r}(\lambda)|^2 dr\Big) d\sigma_\Lambda(\lambda)\,.$$

Fixing $\lambda > 0$, one computes $\widehat{\chi}_{B_r}(\lambda) = (r/\lambda)^{n/2} J_{n/2}(\lambda r)$ and show that

$$\frac{1}{R\lambda^n}\int_0^R r^n J_{n/2}(\lambda r)^2 dr \asymp_n \frac{R^{n-1}}{\lambda^{n+1}}\,, \quad R \gg 0\,.$$

This asymptotic is uniform in $\lambda \in [\lambda_o, +\infty)$ and $R \geq R_o$ for $\lambda_o > R_o^{-1}$, so we can choose the lower bound

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda \cap B_r))dr \gg_n \Big(\int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}}\Big) R^{n-1}\,.$$

Idea of proof for Beck's Theorem

We have

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda \cap B_r))dr = \int_0^\infty \Big(\frac{1}{R}\int_0^R |\widehat{\chi}_{B_r}(\lambda)|^2 dr\Big) d\sigma_\Lambda(\lambda)\,.$$

Fixing $\lambda > 0$, one computes $\widehat{\chi}_{B_r}(\lambda) = (r/\lambda)^{n/2} J_{n/2}(\lambda r)$ and show that

$$\frac{1}{R\lambda^n}\int_0^R r^n J_{n/2}(\lambda r)^2 dr \asymp_n \frac{R^{n-1}}{\lambda^{n+1}}\,, \quad R \gg 0\,.$$

This asymptotic is uniform in $\lambda \in [\lambda_o, +\infty)$ and $R \geq R_o$ for $\lambda_o > R_o^{-1}$, so we can choose the lower bound

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda \cap B_r))dr \gg_n \Big(\int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}}\Big) R^{n-1}\,.$$

Idea of proof for Beck's Theorem

We have

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda\cap B_r))dr = \int_0^\infty \Big(\frac{1}{R}\int_0^R |\widehat{\chi}_{B_r}(\lambda)|^2 dr\Big) d\sigma_\Lambda(\lambda)\,.$$

Fixing $\lambda > 0$, one computes $\widehat{\chi}_{B_r}(\lambda) = (r/\lambda)^{n/2} J_{n/2}(\lambda r)$ and show that

$$\frac{1}{R\lambda^n}\int_0^R r^n J_{n/2}(\lambda r)^2 dr \asymp_n \frac{R^{n-1}}{\lambda^{n+1}}\,, \quad R \gg 0\,.$$

This asymptotic is uniform in $\lambda \in [\lambda_o, +\infty)$ and $R \geq R_o$ for $\lambda_o > R_o^{-1}$, so we can choose the lower bound

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda\cap B_r))dr \gg_n \Big(\int_{\lambda_o}^\infty \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}}\Big) R^{n-1}\,.$$

There are two things to note:

1. If there is a $\lambda_o > 0$ such that $\sigma_\Lambda([0, \lambda_o]) = 0$, then

$$\frac{1}{R}\int_0^R \mathrm{Var}(\#(\Lambda \cap B_r))dr \asymp_n \Big(\int_{\lambda_o}^{\infty} \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}}\Big) R^{n-1}\,.$$

   We call such point processes Λ ***stealthy***. (For example Λ_{Z^n}.)

2. The number variance might behave differently along specfic subsequences. We show that there is a sequence $R_j \to +\infty$ such that

$$\mathrm{Var}(\#(\Lambda_{\mathsf{Z}^5} \cap B_{R_j})) = o(R_j^4)\,.$$

Hyperuniformity

From the proof sketch of Beck's Theorem, we are left with investigating integrals of the form

$$\int_0^{\lambda_o} \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}}, \quad \lambda_o \approx 0.$$

Definition

An invariant point process Λ is *spectrally hyperuniform* if

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda([0,\varepsilon])}{\varepsilon^n} = 0.$$

Theorem (Björklund-Hartnick, '22)

Let Λ be an invariant point process in $\mathbb{R}^d$. Then

$$\limsup_{R \to +\infty} \frac{\mathrm{Var}(\#(\Lambda \cap B_R))}{R^n} = 0 \iff \limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda([0,\varepsilon])}{\varepsilon^n} = 0.$$

Examples include lattices, arithmetic model sets, certain determinantal point processes and more.

Hyperuniformity

From the proof sketch of Beck's Theorem, we are left with investigating integrals of the form

$$\int_0^{\lambda_o} \frac{d\sigma_\Lambda(\lambda)}{\lambda^{n+1}}, \quad \lambda_o \approx 0\,.$$

Definition

An invariant point process Λ is *spectrally hyperuniform* if

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda([0,\varepsilon])}{\varepsilon^n} = 0\,.$$

Theorem (Björklund-Hartnick, '22)

Let Λ be an invariant point process in $\mathbb{R}^d$. Then

$$\limsup_{R \to +\infty} \frac{\mathrm{Var}(\#(\Lambda \cap B_R))}{R^n} = 0 \quad \Longleftrightarrow \quad \limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda([0,\varepsilon])}{\varepsilon^n} = 0\,.$$

Examples include lattices, arithmetic model sets, certain determinantal point processes and more.

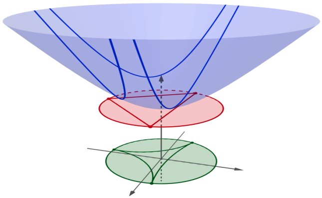
What about other geometries?

Hyperbolic space

Consider the the one-sheeted hyperboloid

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : [x,x] = x_0^2 - x_1^2 - \dots - x_n^2 = 1,\ x_0 > 0 \right\} \cong \mathrm{SO}^+(1,n)/\mathrm{SO}(n)\,.$$

with the metric $d(x,y) = \operatorname{arccosh}([x,y])$.



The $\mathrm{SO}^+(1,n)$-invariant reference measure is in polar/Cartan coordinates given by

$$dm_n(\cosh(t), \sinh(t)v) = \sinh(t)^{n-1} dt\, d\varsigma_{n-1}(v)\,,$$

and the volume growth of the centered $\mathrm{SO}(n)$-invariant ball $B_R = B_R(o)$ is

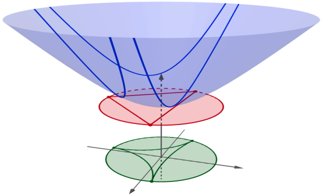
$$\mathrm{Vol}_{\mathbb{H}^n}(B_R) = \varsigma_{n-1}(\mathbb{S}^{n-1}) \int_0^R \sinh(t)^{n-1} dt \asymp \mathrm{e}^{(n-1)R}\,.$$

Hyperbolic space

Consider the the one-sheeted hyperboloid

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : [x,x] = x_0^2 - x_1^2 - \ldots - x_n^2 = 1,\ x_0 > 0 \right\} \cong \mathrm{SO}^+(1,n)/\mathrm{SO}(n)\,.$$

with the metric $d(x,y) = \operatorname{arccosh}([x,y])$.



The $\mathrm{SO}^+(1,n)$-invariant reference measure is in polar/Cartan coordinates given by

$$dm_n(\cosh(t), \sinh(t)v) = \sinh(t)^{n-1} dt\, d\varsigma_{n-1}(v)\,,$$

and the volume growth of the centered $\mathrm{SO}(n)$-invariant ball $B_R = B_R(o)$ is

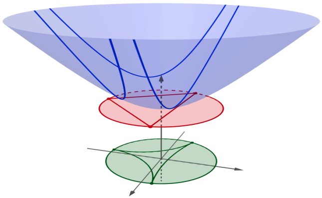
$$\mathrm{Vol}_{\mathbb{H}^n}(B_R) = \varsigma_{n-1}(\mathbb{S}^{n-1}) \int_0^R \sinh(t)^{n-1} dt \asymp \mathrm{e}^{(n-1)R}\,.$$

Hyperbolic space

Consider the the one-sheeted hyperboloid

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} : [x,x] = x_0^2 - x_1^2 - \dots - x_n^2 = 1,\ x_0 > 0 \right\} \cong \mathrm{SO}^+(1,n)/\mathrm{SO}(n)\,.$$

with the metric $d(x,y) = \operatorname{arccosh}([x,y])$.



The $\mathrm{SO}^+(1,n)$-invariant reference measure is in polar/Cartan coordinates given by

$$dm_n(\cosh(t), \sinh(t)v) = \sinh(t)^{n-1} dt\, d\varsigma_{n-1}(v)\,,$$

and the volume growth of the centered $\mathrm{SO}(n)$-invariant ball $B_R = B_R(o)$ is

$$\mathrm{Vol}_{\mathsf{H}^n}(B_R) = \varsigma_{n-1}(\mathbb{S}^{n-1}) \int_0^R \sinh(t)^{n-1} dt \asymp \mathrm{e}^{(n-1)R}\,.$$

Fourier analysis on $\mathbb{H}^d$

We again consider the radial Fourier transform, which for $\mathbb{H}^n$ is

$$\widehat{f}(\lambda) = \int_{\mathsf{H}^n} f(x)\phi_\lambda(\ell(x))dm_n(x)\,, \quad f \in C_c(\mathbb{H}^n) \text{ radial},$$

where $\ell(x) = d(x, o)$ and

$$\phi_\lambda(\ell(x)) = \text{const. } \sinh(\ell(x))^{2-n} \int_0^{\ell(x)} (\cosh(\ell(x)) - \cosh(s))^{\frac{n-3}{2}} \cos(\lambda s) ds$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{H}^n$.

Here we allow for $\lambda \in [0, +\infty) \cup i[0, (n-1)/2]$.

There is moreover a Plancherel formula

$$\int_{\mathsf{H}^n} |f(x)|^2 dm_n(x) = \int_0^\infty |\widehat{f}(\lambda)|^2 \frac{d\lambda}{|c_n(\lambda)|^2}$$

where $c_n(\lambda) \propto \Gamma(i\lambda)/\Gamma(i\lambda + (n-1)/2)$ is the *Harish-Chandra c-function*.

We again consider the radial Fourier transform, which for $\mathbb{H}^n$ is

$$\widehat{f}(\lambda) = \int_{\mathsf{H}^n} f(x)\phi_\lambda(\ell(x))dm_n(x)\,, \qquad f \in C_c(\mathbb{H}^n) \text{ radial},$$

where $\ell(x) = d(x, o)$ and

$$\phi_\lambda(\ell(x)) = \text{const. } \sinh(\ell(x))^{2-n} \int_0^{\ell(x)} (\cosh(\ell(x)) - \cosh(s))^{\frac{n-3}{2}} \cos(\lambda s) ds$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{H}^n$.

Here we allow for $\lambda \in [0, +\infty) \cup i[0, (n-1)/2]$.

There is moreover a Plancherel formula

$$\int_{\mathsf{H}^n} |f(x)|^2 dm_n(x) = \int_0^\infty |\widehat{f}(\lambda)|^2 \frac{d\lambda}{|c_n(\lambda)|^2}$$

where $c_n(\lambda) \propto \Gamma(i\lambda)/\Gamma(i\lambda + (n-1)/2)$ is the *Harish-Chandra c-function*.

We again consider the radial Fourier transform, which for $\mathbb{H}^n$ is

$$\widehat{f}(\lambda) = \int_{\mathsf{H}^n} f(x)\phi_\lambda(\ell(x))dm_n(x)\,, \quad f \in C_c(\mathbb{H}^n) \text{ radial},$$

where $\ell(x) = d(x, o)$ and

$$\phi_\lambda(\ell(x)) = \text{const. } \sinh(\ell(x))^{2-n} \int_0^{\ell(x)} (\cosh(\ell(x)) - \cosh(s))^{\frac{n-3}{2}} \cos(\lambda s) ds$$

are the bounded eigenfunctions of the radial Laplacian on $\mathbb{H}^n$.

Here we allow for $\lambda \in [0, +\infty) \cup i[0, (n-1)/2]$.

There is moreover a Plancherel formula

$$\int_{\mathsf{H}^n} |f(x)|^2 dm_n(x) = \int_0^\infty |\widehat{f}(\lambda)|^2 \frac{d\lambda}{|c_n(\lambda)|^2}$$

where $c_n(\lambda) \propto \Gamma(i\lambda)/\Gamma(i\lambda + (n-1)/2)$ is the *Harish-Chandra c-function*.

Hyperbolic diffraction

The existence of a diffraction measure for invariant point processes is slightly more subtle for $\mathbb{H}^n$.

Theorem (Krein, Gelfand-Vilenkin)

Let Λ be an invariant point process on $\mathbb{H}^n$. Then there is a unique positive Radon measure $\sigma_\Lambda^{(p)}$ on $(0,+\infty)$ and a unique positive finite measure $\sigma_\Lambda^{(c)}$ on $[0,(n-1)/2)$ such that

$$\operatorname{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda^{(p)}(\lambda) + \int_0^{\frac{n-1}{2}} |\widehat{f}(i\lambda)|^2 d\sigma_\Lambda^{(c)}(\lambda)\,.$$

We refer to $\sigma_\Lambda^{(p)}, \sigma_\Lambda^{(c)}$ as the *principal* and *complementary di raction measures* of Λ respectively.

Example (Hyperbolic Poisson diffraction)

The diffraction of the Vol_d-Poisson point process is $d\sigma_{\mathrm{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma_\Lambda^{(c)}(\lambda) = 0$ by the Plancherel formula.

Hyperbolic diffraction

The existence of a diffraction measure for invariant point processes is slightly more subtle for $\mathbb{H}^n$.

Theorem (Krein, Gelfand-Vilenkin)

Let Λ be an invariant point process on $\mathbb{H}^n$. Then there is a unique positive Radon measure $\sigma_\Lambda^{(p)}$ on $(0,+\infty)$ and a unique positive finite measure $\sigma_\Lambda^{(c)}$ on $[0,(n-1)/2)$ such that

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda^{(p)}(\lambda) + \int_0^{\frac{n-1}{2}} |\widehat{f}(i\lambda)|^2 d\sigma_\Lambda^{(c)}(\lambda)\,.$$

We refer to $\sigma_\Lambda^{(p)}, \sigma_\Lambda^{(c)}$ as the *principal* and *complementary di raction measures* of Λ respectively.

Example (Hyperbolic Poisson diffraction)

The diffraction of the Vol_d-Poisson point process is $d\sigma_{\mathrm{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma_\Lambda^{(c)}(\lambda) = 0$ by the Plancherel formula.

Hyperbolic diffraction

The existence of a diffraction measure for invariant point processes is slightly more subtle for $\mathbb{H}^n$.

Theorem (Krein, Gelfand-Vilenkin)

Let Λ *be an invariant point process on* $\mathbb{H}^n$. *Then there is a unique positive Radon measure* $\sigma_\Lambda^{(p)}$ *on* $(0, +\infty)$ *and a unique positive finite measure* $\sigma_\Lambda^{(c)}$ *on* $[0, (n-1)/2)$ *such that*

$$\mathrm{Var}\Big(\sum_{p\in\Lambda} f(p)\Big) = \int_0^\infty |\widehat{f}(\lambda)|^2 d\sigma_\Lambda^{(p)}(\lambda) + \int_0^{\frac{n-1}{2}} |\widehat{f}(i\lambda)|^2 d\sigma_\Lambda^{(c)}(\lambda)\,.$$

We refer to $\sigma_\Lambda^{(p)}, \sigma_\Lambda^{(c)}$ as the *principal* and *complementary di raction measures* of Λ respectively.

Example (Hyperbolic Poisson diffraction)

The diffraction of the Vol_d-Poisson point process is $d\sigma_{\mathrm{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma_\Lambda^{(c)}(\lambda) = 0$ by the Plancherel formula.

An analogue of Beck's Theorem

Theorem (Björklund-B.)

Let Λ be an invariant point process on $\mathbb{H}^d$.

1. *If $\sigma_\Lambda^{(c)} \neq 0$ then*

$$\frac{\mathrm{Var}(\#(\Lambda \cap B_R))}{\mathrm{e}^{(n-1)R}} \underset{R\to+\infty}{\longrightarrow} +\infty$$

2. *If $\sigma_\Lambda^{(c)} = 0$ then*

$$\lim_{R\to+\infty} \frac{1}{R} \int_0^R \frac{\mathrm{Var}(\#(\Lambda \cap B_r))}{\mathrm{e}^{(n-1)r}} dr \gg_n \int_{\lambda_o}^{\infty} |c_{n+2}(\lambda)|^2 d\sigma_\Lambda^{(p)}(\lambda) > 0\,.$$

In particular, Λ is not "geometrically hyperuniform".

Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0,\varepsilon])$ with $\sigma_{\text{Poi}}([0,\varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma_{\text{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma_{\text{Poi}}^{(c)} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma_{\text{Poi}}^{(p)}([0,\varepsilon]) \approx \varepsilon^3 .$$

Definition

An invariant point process Λ on $\mathbb{H}^n$ is *spectrally hyperuniform* if $\sigma_\Lambda^{(c)} = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda^{(p)}([0,\varepsilon])}{\varepsilon^3} = 0 .$$

Geometrically, we compared the number variance with

$$\text{Vol}_{\mathbb{H}^n}(B_R) = \widehat{\chi}_{B_R}\left(i\frac{n-1}{2}\right),$$

but $\lambda = i\frac{n-1}{2}$ does **not** lie in the support of $\sigma_{\text{Poi}}^{(p)}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$

Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0,\varepsilon])$ with $\sigma_{\mathrm{Poi}}([0,\varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma^{(p)}_{\mathrm{Poi}}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma^{(c)}_{\mathrm{Poi}} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma^{(p)}_{\mathrm{Poi}}([0,\varepsilon]) \approx \varepsilon^3 \,.$$

Definition

An invariant point process Λ on $\mathbb{H}^n$ is *spectrally hyperuniform* if $\sigma^{(c)}_\Lambda = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma^{(p)}_\Lambda([0,\varepsilon])}{\varepsilon^3} = 0 \,.$$

Geometrically, we compared the number variance with

$$\mathrm{Vol}_{\mathbb{H}^n}(B_R) = \widehat{\chi}_{B_R}\left(i\frac{n-1}{2}\right),$$

but $\lambda = i\frac{n-1}{2}$ does **not** lie in the support of $\sigma^{(p)}_{\mathrm{Poi}}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$

Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0,\varepsilon])$ with $\sigma_{\mathrm{Poi}}([0,\varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma^{(p)}_{\mathrm{Poi}}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma^{(c)}_{\mathrm{Poi}} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma^{(p)}_{\mathrm{Poi}}([0,\varepsilon]) \approx \varepsilon^3 .$$

Definition

An invariant point process Λ on $\mathbb{H}^n$ is *spectrally hyperuniform* if $\sigma^{(c)}_\Lambda = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma^{(p)}_\Lambda([0,\varepsilon])}{\varepsilon^3} = 0 .$$

Geometrically, we compared the number variance with

$$\mathrm{Vol}_{\mathsf{H}^n}(B_R) = \widehat{\chi}_{B_R}\left(i\frac{n-1}{2}\right),$$

but $\lambda = i\frac{n-1}{2}$ does **not** lie in the support of $\sigma^{(p)}_{\mathrm{Poi}}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$

Spectral hyperuniformity

Euclidean spectral hyperuniformity: Compare $\sigma_\Lambda([0,\varepsilon])$ with $\sigma_{\mathrm{Poi}}([0,\varepsilon]) \propto \varepsilon^n$.

For hyperbolic space $\mathbb{H}^n$ we had $d\sigma_{\mathrm{Poi}}^{(p)}(\lambda) = |c_n(\lambda)|^{-2} d\lambda$ and $d\sigma_{\mathrm{Poi}}^{(c)} = 0$. For small $\varepsilon > 0$, $|c_n(\varepsilon)|^{-2} \approx \varepsilon^2$, so

$$\sigma_{\mathrm{Poi}}^{(p)}([0,\varepsilon]) \approx \varepsilon^3 .$$

Definition

An invariant point process Λ on $\mathbb{H}^n$ is *spectrally hyperuniform* if $\sigma_\Lambda^{(c)} = 0$ and

$$\limsup_{\varepsilon \to 0^+} \frac{\sigma_\Lambda^{(p)}([0,\varepsilon])}{\varepsilon^3} = 0 .$$

Geometrically, we compared the number variance with

$$\mathrm{Vol}_{\mathsf{H}^n}(B_R) = \widehat{\chi}_{B_R}(i\frac{n-1}{2}) ,$$

but $\lambda = i\frac{n-1}{2}$ does **not** lie in the support of $\sigma_{\mathrm{Poi}}^{(p)}$.

Geometric and spectral hyperuniformity are not equivalent for $\mathbb{H}^n$

What is the ”geometric” counterpart of spectral hyperuniformity?

Heat kernel hyperuniformity

It is in general quite hard to compute diffraction measures of point processes. We consider the ***heat kernel***

$$h_t(x) = \int_0^\infty \mathrm{e}^{-t((\frac{n-1}{2})^2+\lambda^2)} \phi_\lambda(\ell(x)) \frac{d\lambda}{|c_n(\lambda)|^2}, \quad t > 0.$$

Theorem (Björklund-B., in the works)

An invariant point process Λ on $\mathbb{H}^n$ is spectrally hyperuniform if and only if

$$\limsup_{t\to+\infty} t^{3/2} \mathrm{e}^{(n-1)^2 t/2} \mathrm{Var}\Big(\sum_{p\in\Lambda} h_t(p)\Big) = 0.$$

Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?
- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?
- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?
- Is there a similar notion of "order"/"uniformity" for processes admitting complementary spectrum?
- What point processes are stealthy?

Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?
- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?
- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?
- Is there a similar notion of "order"/"uniformity" for processes admitting complementary spectrum?
- What point processes are stealthy?

Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?
- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?
- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?
- Is there a similar notion of "order"/"uniformity" for processes admitting complementary spectrum?
- What point processes are stealthy?

Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?
- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?
- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?
- Is there a similar notion of "order"/"uniformity" for processes admitting complementary spectrum?
- What point processes are stealthy?

Questions

- What are examples of heat kernel/spectrally hyperuniform point processes in hyperbolic space?
- Other geometries: Heisenberg motion groups, automorphism groups of trees. Easier to find examples?
- Is the failure of geometric hyperuniformity on $\mathbb{H}^n$ a strict consequence of non-amenability?
- Is there a similar notion of "order"/"uniformity" for processes admitting complementary spectrum?
- What point processes are stealthy?

Thank you!

Sources

1. J. Beck, "Irregularities of distribution", ACTA matematica vol. 159, 1987:

   https://projecteuclid.org/journals/acta-mathematica/volume-159/issue-none/Irregularities-of-distribution-I/10.1007/BF02392553.full

2. M. Björklund, T. Hartnick, "Hyperuniformity and non-hyperuniformity of quasicrystals", 2022:

   https://arxiv.org/abs/2210.02151

3. F. Jenni, "Ueber den ersten Eigenwert des Laplace-Operators auf ausgewählten Beispelen kompakter Riemannscher Flächen", Comment. Math. Helvetici, vol. 59, 1984:

   https://gdz.sub.uni-goettingen.de/id/PPN358147735_0059?tify=%7B%22pages%22%3A%5B197%5D%2C%22pan%22%3A%7B%22x%22%3A0.847%2C%22y%22%3A0.788%7D%2C%22view%22%3A%22info%22%2C%22zoom%22%3A0.352%7D