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Point processes and model sets

Workshop Aperiodic order and approximate groups

Maximilian Wackenhuth | 23. August 2023

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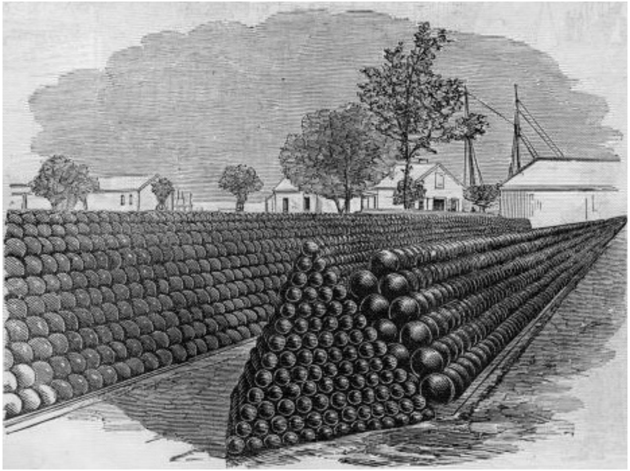

Sphere packings

Sphere packing

A sphere packing is a set of disjoint open balls of equal radius.

Classical density

The classical density of a sphere packing S is given by

$$D(S,x) := \lim_{R\to\infty} \frac{\mathrm{vol}(B(x,R)\cap\bigcup S)}{\mathrm{vol}(B(x,R))}$$

- Böröczky: Classical density very degenerate in $\mathbb{H}^n$
- Bowen: Study “random sphere packings” to filter out Böröczkys degenerate examples.
- Bowen: For only countably many radii periodic sphere packings in $\mathbb{H}^n$ are optimally dense.

Setting

Setting

- G lcsc group.
- $K \leq G$ compact.
- G/K has G-invariant metric d.
- Nice pointwise ergodic theorem for G:

$$\frac{1}{m_G(G_t)} \int_{G_t} f(g^{-1}.x)dm_G(g) \to \int_X fd\mu, \quad \forall f \in C_c(X)$$

holds for all x in G-invariant set of full measure.

Examples

- $\mathbb{R}^n$: $G = \mathbb{R}^n$, $K = \{e\}$.
- $\mathbb{R}^n$: $G = \mathrm{Iso}(\mathbb{R}^n)$, $K = O(n)$.
- $\mathbb{H}^2$: $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$.
- $\mathbb{H}^n$: $G = \mathrm{SO}(n,1)$, $K = \mathrm{O}(n)$.

Point processes

Example: Lattices

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$\Gamma \leq G$ lattice (discrete and cofinite subgroup). $\Omega_\Gamma := G/\Gamma$

$$\xi : (\Omega_\Gamma, \mathbb{P}) \to \mathrm{Cl}(G), \quad g\Gamma \mapsto g\Gamma$$

$\rightsquigarrow$ “random” translate of lattice Γ.

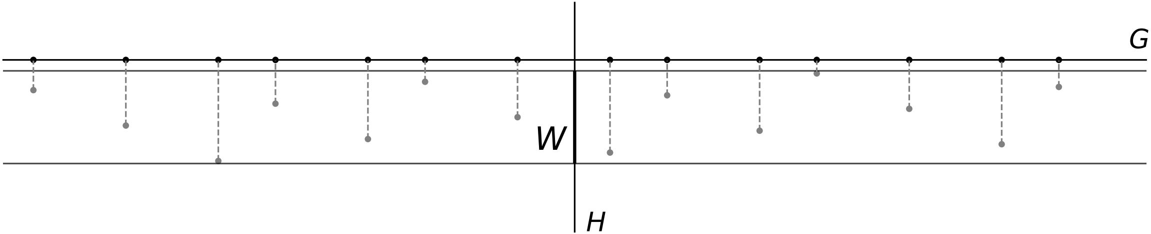
Example: Model sets

Model sets

- H lcsc group.
- $\Gamma < G \times H$ a lattice projecting densely to H and injectively to G.
- $W \subset H$ compact.

Then the set $\pi_G(G \times W \cap \Gamma) \subset G$ is called a *model set*.

Example: Model sets



Example: Model sets

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Λ model set.

$$\Omega_\Lambda := \overline{G.\Lambda} \setminus \{\emptyset\}$$

W nice $\implies$ exists unique G-inv. probability measure $\mathbb{P}$ on Ω_Λ.

$$\xi : (\Omega_\Lambda, \mathbb{P}) \to \mathrm{Cl}(G), \quad \Lambda' \mapsto \Lambda'$$

$\rightsquigarrow$ “random” uniformly discrete point set.

Example: Approximate lattices

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Approximate groups

$\Lambda \subset G$ such that

1. $e \in \Lambda$
2. $\Lambda = \Lambda^{-1}$,
3. there is a finite set F such that $\Lambda^2 \subset F\Lambda$

Example: Approximate lattices

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Strong approximate lattices

$\Lambda \subset G$ uniformly discrete approximate group s.t. $\exists$ nontrivial G-invariant measure $\mathbb{P}$ on $\Omega_\Lambda := \overline{G.\Lambda} \setminus \{\emptyset\}$.

$\rightsquigarrow \xi : (\Omega_\Lambda, \mathbb{P}) \to \mathrm{Cl}(G), \Lambda' \to \Lambda'$ "random" point set in G.

Point processes

Point processes

$(\Omega, \mathbb{P})$ probability space.

$$\xi : (\Omega, \mathbb{P}) \to \mathrm{Cl}(G/K)$$

s.t.

- $\xi(\omega) \subset G/K$ countable for almost all $\omega \in \Omega$
- ξ measurable.

$\mu_\xi := \xi_* \mathbb{P}$ is called distribution of ξ.

- ξ uniformly discrete/FLC/... $:\Leftrightarrow$ $\xi(\omega)$ is uniformly discrete/FLC/... for almost all ω
- Call ξ ergodic if μ_ξ is G-ergodic.

In all of the previous examples $K = \{e\}$.

Associated objects

Hof autocorrelation

ξ uniformly discrete ergodic point process.

Hof autocorrelation

Fix a “generic” point Λ of ξ.

$$\eta_\xi(f) := \lim_{R\to\infty} \frac{1}{\operatorname{vol}(B(x_0, R))} \sum_{x\in\Lambda\cap B(x_0,R)} \sum_{y\in\Lambda} f(x^{-1}y), \quad \forall f \in C_c(K\backslash G/K).$$

In many cases this gives a well-defined measure on G/K only depending on ξ.

Autocorrelation measure

The Hof autocorrelation is conceptually easy, but there is a measure that is easier to work with.

Periodization

ξ ergodic uniformly discrete point process, Λ generic point of ξ.

$$\implies \Omega_\Lambda := \overline{G.\Lambda} \setminus \{\emptyset\} \text{ has full measure wrt. } \mu_\xi.$$

Periodization

$$\mathcal{P}_\xi : C_c(G/K) \to C_0(\Omega_\Lambda), f \mapsto (\Lambda' \mapsto \sum_{x \in \Lambda'} f(x)).$$

Autocorrelation of point processes

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$$\eta_\xi(f^* * g) = \int_{\Omega_\Lambda} \overline{\mathcal{P}_\xi(f)}\mathcal{P}_\xi(g)d\mu_\xi = \mathbb{E}[\overline{\mathcal{P}_\xi(f)}\mathcal{P}_\xi(g)], \quad \forall f, g \in C_c(G/K)$$

Hof’s autocorrelation measure and the autocorrelation

If G has a “nice” ergodic theorem, then the autocorrelation measure and Hof’s autocorrelation are equal.

The intensity

$$\int \mathcal{P}_\xi(f) d\mu_\xi = i(\xi) \int f dm_{G/K}, \quad \forall f \in C_c(G/K)$$

Example

- ξ constructed from lattice: $i(\xi) = \frac{1}{|\Gamma|}$
- ξ constructed from model set: $i(\xi) = \frac{m_H(W)}{|\Gamma|}$

Diffraction

Diffraction

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From now on: $C_c(K\backslash G/K)$ commutative.

Diffraction/Spherical diffraction

The *diffraction* of the point process ξ is the Fourier transform $\hat{\eta}_\xi$ of the autocorrelation measure.

Appropriate notion of Fourier transform: spherical transform.

Defining property:

$$\eta_\xi(f^* * f) = \widehat{\eta}_\xi(\widehat{f^* * f}), \quad \forall f \in C_c(K\backslash G/K).$$

Properties of diffraction

- η_ξ is positive measure.
- $\eta_\xi(\{\omega_0\}) = i(\xi)^2$, where ω_0 denotes the "trivial spherical function".
- $\eta_\xi(f^* * f) = \widehat{\eta}_\xi(\widehat{f^* * f})$ for all $f \in C_c(K\backslash G/K)$.

Sphere packings

Classical density is not well-behaved.

- Classical density is not easy to work with, even in euclidean space there are issues with oscillation.
- Existence of packings that maximize classical density was show by Groemer in 1961.
- In hyperbolic space exponential volume growth leads to dominating boundary terms and dependence on the point x that do not appear in the euclidean case.
- Examples by Böröczky show that the notion of density in the classical sense is very degenerate in hyperbolic space.

Sphere packings

We can think of r-uniformly discrete sets as sphere packings by spheres of radius r.
$\rightsquigarrow$ Think of r-uniformly discrete point processes as random sphere packings.

Sphere packings

Bowen and Radin: Define density as

$$D(\xi) = \mathbb{P}(d(\xi, x_0) < r),$$

Pointwise ergodic theorems $\implies$

$$D(\xi) = \text{ classical density of } \Lambda \text{ for each generic point } \Lambda \text{ of } \xi.$$

if ξ ergodic.

Sphere packings

Bowen and Radin:
Finding maximal sphere packing density $\Longleftrightarrow$ finding maximal packing density of ergodic random sphere packinga.

Density of random sphere packings

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Density formula (W.)

$$D(\xi) = m_{G/K}(B(eK, r))i(\xi).$$

Hence estimating the maximal intensity of point processes in G/K is a worthwhile goal.

Cohn-Elkies argument

Cohn-Elkies estimate

Cohn and Elkies obtained the following bounds for the density of sphere packings in $\mathbb{R}^n$:

$$C\frac{f(0)}{\widehat{f}(0)}$$

for nice f such that $\widehat{f} \geq 0$, $\widehat{f}$ and $f(x) \leq 0$ for $\|x\| > r$.
$\rightsquigarrow$ Reinterpret their proof in the language of point processes and generalize to get bounds for $i(\xi)$.

$\Gamma \leq \mathbb{R}^n$ lattice.

$$\sum_{x\in\Gamma} f(x) = \frac{1}{|\Gamma|} \sum_{t\in\Gamma^*} \widehat{f}(t),$$

$$\sum_{x\in\Gamma} f(x) \leq f(0) \quad \text{and} \quad \frac{1}{|\Gamma|} \sum_{t\in\Gamma^*} \widehat{f}(t) \geq \frac{1}{|\Gamma|}\widehat{f}(0),$$

$$\implies f(0) \geq \frac{1}{|\Gamma|}\widehat{f}(0)$$

Conjecture by Cohn and Zhao: This bound holds in hyperbolic space.

Cohn, Lurie and Sarnak: Bound valid for periodic packings.

Argument for point processes

$$\lim_{R\to\infty} \frac{1}{m_{G/K}(B(eK,R))} \sum_{x\in\Lambda\cap B(eK,R)} \sum_{y\in\Lambda)} f(x^{-1}y) = i(\xi)^2\widehat{f}(\omega_0) + (\widehat{\eta}_\xi(\widehat{f}) - i(\xi)^2\widehat{f}(\omega_0))$$

$$i(\xi)^2\widehat{f}(\omega_0) + (\widehat{\eta}_\xi(\widehat{f}) - i(\xi)^2\widehat{f}(\omega_0)) \geq i(\xi)^2\widehat{f}(\omega_0)$$

$$\lim_{R\to\infty} \frac{1}{m_{G/K}(B(eK,R))} \sum_{x\in\Lambda\cap B(eK,R)} \sum_{y\in\Lambda} f(x^{-1}y) \leq \lim_{R\to\infty} \frac{1}{m_{G/K}(B(eK,R))} \sum_{x\in\Lambda\cap B(eK,R)} f(KeK) = i(\xi)f(KeK)$$

Bound for point processes

$$i(\xi) \leq \frac{f(KeK)}{\widehat{f}(\omega_0)}$$