Bad variances

MB
Cut and project processes

Let \((\mathbb{R}^d, \mathbb{R}^q, \Gamma, W)\) be a cut-and-project scheme, i.e.

- \(\Gamma < \mathbb{R}^d \times \mathbb{R}^q\) is a lattice such that \(\text{pr}_{\mathbb{R}^d}|_{\Gamma}\) is injective and \(\text{pr}_{\mathbb{R}^q}(\Gamma)\) is dense.
- \(W \subset \mathbb{R}^q\) is a bounded Borel set with non-empty interior.

For \((x, y) \in \mathbb{R}^d \times \mathbb{R}^q\), define the cut-and-project set \(P(x, y) = \text{pr}_{\mathbb{R}^d}((\Gamma + (x, y)) \cap (\mathbb{R}^d \times W)) \subset \mathbb{R}^d\).

The first bullet ensures that \(\Gamma + (x, y) \mapsto P(x, y)\) defines a simple and ergodic point process in \(\mathbb{R}^d\).

The second bullet ensures that \(P(x, y)\) is a relatively dense and uniformly discrete set in \(\mathbb{R}^d\) for every \((x, y)\).
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For \((x, y) \in \mathbb{R}^d \times \mathbb{R}^q\), define the \textbf{cut-and-project set}

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P_{(x,y)} = \text{pr}_{\mathbb{R}^d}(((\Gamma + (x, y)) \cap (\mathbb{R}^d \times W)) \subset \mathbb{R}^d).
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Let \((\mathbb{R}^d, \mathbb{R}^q, \Gamma, \mathcal{W})\) be a cut-and-project scheme, i.e.

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- \(\mathcal{W} \subset \mathbb{R}^q\) is a bounded Borel set with non-empty interior.

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The number variance

Let $m$ denote the unique $R^d \times R^q$-invariant probability measure on the torus $\Gamma \setminus R^d \times R^q$. Note that

$$Z_{\Gamma \setminus R^d \times R^q} |P(x, y) \cap B| dm(x, y) = \iota_W \text{Vol}^d(B),$$

for every bounded Borel set $B \subset R^d$, where $\iota_W = \text{Vol}^q(W)$ $\text{Covol}(\Gamma)$.

Let $B_{R}$ denote the centered Euclidean ball of radius $R$ and define the discrepancy $D_R(x, y) = |P(x, y) \cap B_{R}| - \iota_W \text{Vol}^d(B_{R})$ and the number variance $N_R(\Gamma, W) := \int_{\Gamma \setminus R^d \times R^q} |D_R|^2 dm$. 

The number variance

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\int_{\Gamma \setminus \mathbb{R}^d \times \mathbb{R}^q} |P(x, y) \cap B| \, dm(x, y) = \nu_W \, \text{Vol}_d(B),
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for every bounded Borel set \( B \subset \mathbb{R}^d \), where \( \nu_W = \frac{\text{Vol}_q(W)}{\text{Covol}(\Gamma)} \).
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and the number variance

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N_R(\Gamma, W) := \int_{\Gamma \backslash \mathbb{R}^d \times \mathbb{R}^q} |D_R|^2 \, dm.
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The mean ergodic theorem tells us that
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\lim_{R \to \infty} N_R(\Gamma, W) R^d = 0.
\]
A theorem of Beck tells us that
\[
\limsup_{R \to \infty} N_R(\Gamma, W) R^d - 1 > 0.
\]
Definition \((\Gamma, W)\) is hyperuniform if
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Hyperuniformity

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Definition

(\(\Gamma, W\)) is **hyperuniform** if

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Repellence and hyperuniformity

Definition

Let \( \beta > 0 \). A lattice \( \Gamma \subset \mathbb{R}^d \times \mathbb{R}^q \) is \( \beta \)-repellent if for all sufficiently \( \epsilon > 0 \), and for all \((\gamma_1, \gamma_2) \in \Gamma \setminus \{0, 0\}\)
\[
\|\gamma_1\| < \epsilon \implies \|\gamma_2\| \geq \epsilon - \beta.
\]

Fact:

Generic lattices and badly approximable lattices (e.g. arithmetic) are \( \beta \)-repellent for some \( \beta > 0 \).

Theorem (B-Hartnick, '22)

If \( \Gamma \perp \) is \( \beta \)-repellent for some \( \beta > d + \delta \), then \( (\Gamma, W) \) is hyperuniform for every "\( \delta \)-nice enough" \( W \) (e.g. Euclidean balls (\( \delta = 1 \)), smooth convex sets with non-vanishing curvatures).
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Let us now introduce a class of highly non-repellent lattices:
Liouvillean lattices and non-hyperuniformity

Let us now introduce a class of highly non-repellent lattices:

**Definition**

Γ < \mathbb{R}^p \times \mathbb{R}^q is **Liouvillean** if there exist a constant \( c > 0 \) and a sequence \( \gamma^{(k)} = (\gamma_1^{(k)}, \gamma_2^{(k)}) \in \Gamma \setminus \{0\} \) such that

- \( \gamma_1^{(k)} \to 0 \), as \( k \to \infty \),
- \( \|\gamma_2^{(k)}\| \leq c\|\gamma_1^{(k)}\|^{-1/k} \), for all \( k \).
An example

Let $a \geq 0$ be a Liouville number, i.e. $a$ is irrational and there is a sequence $(m_k, n_k) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $m_k \to \infty$ and $|am_k - n_k| \leq m_k^{-k}$, for all $k$.

The lattice $\Gamma = \{(am_k - n_k, am_k + n_k) : (m_k, n_k) \in \mathbb{Z}^2\}$ is Liouvillean (with $\gamma(k) = (am_k - n_k, am_k + n_k)$).
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Roundish sets

Definition
A bounded Borel set $W \subset \mathbb{R}^q$ is roundish if there exists $(\alpha, \beta) \in (0, \infty)^2$ such that
$$b\chi_W(y) = \cos\left(\alpha \|y\| - \beta\right) \frac{\|y\|}{d+1/2}.$$ as $y \to \infty$ (plus some minor technical conditions).

Example: If $W = [-1/2, 1/2] \subset \mathbb{R}$, then
$$b\chi_W(y) = \sin\left(\frac{\pi}{\|y\|}\right) = \cos\left(\pi |y| - \frac{\pi}{2}\right) \frac{\|y\|}{\pi|y|}.$$ Every Euclidean ball is roundish by a theorem of Hankel.
Roundish sets

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A bounded Borel set $W \subset \mathbb{R}^q$ is **roundish** if there exists $(\alpha, \beta) \in (0, \infty)^2$ such that

$$\hat{\chi}_W(y) = \frac{\cos(\alpha\|y\| - \beta)}{\|y\|^{(d+1)/2}} + l.o.t.$$

as $y \to \infty$ (±some minor technical conditions).

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Non-hyperuniformity

Theorem (B-Hartnick, ‘22)

Suppose $\Gamma \perp < R^d \times R^q$ is Liouvillean and $W \subset R^q$ is roundish. Then, for Lebesgue almost every $b > 0$, 

$$\lim_{R \to \infty} N_R(\Gamma, bW)_{R^2 - \delta} = \infty,$$

for all $\delta > 0$. In other words, for almost every $b > 0$, the pair $(\Gamma, bW)$ is very far from being hyperuniform.
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The proof

Let \((R^d, R^q, \Gamma, W)\) be an arbitrary cut-and-project scheme.
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Let \((\mathbb{R}^d, \mathbb{R}^q, \Gamma, W)\) be an arbitrary cut-and-project scheme. By Poisson summation,

\[ N_R(\Gamma, W) = c \cdot R^{2d} \cdot \sum_{\xi \in \Gamma^\perp \setminus \{0\}} |\hat{\chi}_B(R\xi_1)|^2 |\chi_W(\xi_2)|^2, \]

where \(B = B_1\) and \(c\) is an explicit constant.
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where \(B = B_1\) and \(c\) is an explicit constant.

Fix \(\varepsilon_0 > 0\) such that \(|\hat{\chi}_B(y)|^2 \geq \frac{1}{2} \text{Vol}_d(B)^2\) for all \(\|y\| \leq \varepsilon_0\). Then, for all \(\delta > 0\) and \(\varepsilon_R < \frac{\varepsilon_0}{R}\), we have

\[
\frac{N_R(\Gamma, bW)}{R^{2d-\delta}} \gg b^{2d} R^\delta \sum_{\|\xi_1\| < \varepsilon_R} |\chi_W(b\xi_2)|^2,
\]

for all \(b > 0\).
The proof

Let us now assume that $\Gamma^\perp$ is Liouvillean and $W$ is roundish, so that

- There is a sequence $\xi(k) = (\xi(k)_1, \xi(k)_2) \in \Gamma^\perp \setminus \{(0, 0)\}$ with $\xi(k)_1 \to 0$ and $\|\xi(k)_2\| \ll \|\xi(k)_1\|^{-1/k}$, for all $k$.

- $|\chi_W(\xi(k)_2)|^2 \approx \cos^2(\alpha b \|\xi(k)_2\| - \beta) \|\xi(k)_2\|^{d+1}$.
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Let us now assume that $\Gamma^\perp$ is Liouvillean and $W$ is roundish, so that

- there is a sequence $\xi^{(k)} = (\xi_1^{(k)}, \xi_2^{(k)}) \in \Gamma^\perp \setminus \{(0,0)\}$ with $\xi_1^{(k)} \to 0$ and $\|\xi_2^{(k)}\| \ll \|\xi_1^{(k)}\|^{-1/k}$, for all $k$.

- $|\chi_W(\xi_2^{(k)})|^2 \gtrsim \frac{\cos^2(\alpha b\|\xi_2^{(k)}\| - \beta)}{\|\xi_2^{(k)}\|^{d+1}}$. 
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Let us now assume that $\Gamma \perp$ is Liouvillean and $W$ is roundish, so that

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- $|\chi_W(\xi_2^{(k)})|^2 \sim \frac{\cos^2(\alpha b \|\xi_2^{(k)}\| - \beta)}{\|\xi_2^{(k)}\|^{d+1}}$

In particular, by choosing $R_k = \|\xi_1^{(k)}\|^{-1}$ and $\varepsilon_{R_k} \asymp 1/R_k$, we get

$$\frac{N_{R_k}(\Gamma, bW)}{R_k^{2d-\delta}} \gg b R_k^\delta \cdot \frac{\cos^2(\alpha b \|\xi_2^{(k)}\| - \beta)}{\|\xi_2^{(k)}\|^{d+1}} \gg b R_k^{\delta-(d+1)/k} \cos^2(\alpha b \|\xi_2^{(k)}\| - \beta).$$
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The problem now is that (potentially)

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However, standard equidistribution theory tells us that for almost every $b$ we can find a sub-sequence $(k_j)$ such that the sequence

$$\alpha b \|\xi_2^{(k_j)}\| - \beta$$

equidistributes (modulo $2\pi$).
The proof

The problem now is that (potentially)

\[ \cos^2(\alpha b\|\xi_2^{(k)}\| - \beta) \ll R_k^{-\delta}. \]

However, standard equidistribution theory tells us that for almost every \( b \) we can find a sub-sequence \((k_j)\) such that the sequence

\[ \alpha b\|\xi_2^{(k_j)}\| - \beta \]

equidistributes (modulo \( 2\pi \)).

In particular, along some further sub-sequence we can ensure that \( \alpha b\|\xi_2^{(k_j)}\| - \beta \to \pi \), and the problem above does not occur, and hence

\[ \frac{N_{R_{k_j}}(\Gamma, bW)}{R_{k_j}^{2d-\delta}} \to \infty, \quad \text{as } j \to \infty. \]