Four appplications of transverse measures to aperiodic order and approximate lattices

Tobias Hartnick (KIT)
based on joint work with Michael Björklund (Chalmers)

Workshop on approximate groups and aperiodic order
Karlsruhe, August 24th 2023

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We will discuss two very abstract notions:

transverse groupoids and transverse measures.

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- Intersection spaces and multiple transverse recurrence
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- Intersection spaces and multiple transverse recurrence
- Aperiodic Siegel transforms
- Local superrigidity

Table of Contents

Reminder of yesterday

Setting

- $G \curvearrowright X$ action of lcsc group on standard Borel space, $Y \subset X$ Borel
- $Y_x = \{g \in G \mid g.x = Y\}$ (hitting times)
- $\Lambda = \bigcup_{y \in Y} Y_y$ (return times)
- $Y \subset X$ separated transversal if

$$G.Y = X \text{ and } \exists\, U \in \mathfrak{U}_e^G : \; \Lambda \cap U = \{e\}.$$

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Transverse point process

If $\mu \in \mathrm{Prob}(X)^G$, then we get a uniformly discrete point process

$$\xi : (X, \mu) \to \mathrm{UD}(G), \quad \xi(x) = Y_x.$$

(transverse point process of the transverse triple (X, μ, Y))

More structures on Y

Let $Y \subset X$ be a separated transversal. The G-action induces:

Transverse orbit relation

$y_1 \sim y_2 \iff \exists g \in G : g.y_1 = y_2$. (countable equivalence relation!)

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Transverse groupoid

- $\mathcal{G}^{(0)} = Y$.
- $\mathcal{G}^{(1)} = \{(y_2, g, y_1) \in Y \times G \times Y \mid g.y_1 = y_2\} \subset Y \times G \times Y$.
- $s(y_2, g, y_1) = y_1$, $r(y_2, g, y_1) = y_2$. (Local homeos: $\mathcal{G}$ is étale.)
- $(y_3, h, y_2) * (y_2, g, y_1) = (y_3, hg, y_1)$.

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- $(y_3, h, y_2) * (y_2, g, y_1) = (y_3, hg, y_1)$.

If all G-stabilizers are trivial (strongly aperiodic) both carry the same information, otherwise $\mathcal{G}$ carries more information.

First examples

Example (Fully periodic case)

$X = \Gamma \backslash G \supset Y = \{\Gamma\}$.

- Hitting/return times: $Y_{\Gamma g} = g\Gamma g^{-1}$, $\Lambda = \Gamma$.
- Transverse relation/transverse groupoid: $\sim$ trivial relation, $\mathcal{G} = \Gamma \rightrightarrows *$.

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Example (Model set case)

$X = \Gamma\backslash(G \times H) \supset Y = \Gamma\backslash(\{e\} \times W) \cong W$

- Hitting times: $Y_{\Gamma(g,h)} = \pi_G(\Gamma(g,h) \cap (G \times W))$ (cut-and-project)
- Return times: $\Lambda = \pi_G(\Gamma \cap (G \times WW^{-1}))$
- Transverse relation/transverse groupoid: orbit relation of Γ_H on W (totally aperiodic and minimal)

A generic example

Example

If $P_0 \subset G$ is uniformly discrete, then $X = \Omega_{P_0}^{\times}$ has a canonical transversal

$$Y^{\mathrm{can}} = \{Q \in X \mid e \in Q\}.$$

Then $Y_P^{\mathrm{can}} = P$ for all $P \in X$ and $\Lambda \subset P_0 P_0^{-1}$.

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$$\mathcal{G}^{(1)} = \{(Qq^{-1}, q, Q) \in Y^{\text{can}} \times G \times Y^{\text{can}} \mid q \in Q\}.$$

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A basis for the Borel structure on Y^{can} is given by the cylinder sets

$$[p, R] = \{Q \in Y^{\mathrm{can}} \mid Q \cap B_R(e) = p\},$$

where $R > 0$ and p is a centered pattern of radius R in P_0.

A very general example

Let Γ be a countable group and let $P_0 \subset R = P_0 P_0^{-1} \subset \Gamma$.
(Think of $\Lambda \subset \Lambda^2 \subset \Lambda^\infty$ for an approximate group $(\Lambda, \Lambda^\infty)$).

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Example

Using only the pair (P_0, R) and the partial multiplication on R, define

$$\mathcal{G}^{(0)} = \{Q \subset R \mid \forall\, F \subset R \text{ finite } \exists\, p \in P_0 : Q \cap F = P_0 p_0^{-1} \cap F\},$$
$$\mathcal{G}^{(1)} = \{(Qq^{-1}, q, Q) \in \mathcal{G}^{(0)} \times R \times \mathcal{G}^{(0)} \mid Q \in \mathcal{G}^{(0)}, q \in Q\}.$$

This is the transverse groupoid of $(X = \overline{\Gamma.P_0} \setminus \{\emptyset\}, Y_{\mathrm{can}})$.

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This is the transverse groupoid of $(X = \overline{\Gamma.P_0} \setminus \{\emptyset\}, Y_{\mathrm{can}})$.

In fact, if $i : R \to G$ is a partial homomorphism with discrete image into any lcsc group G, then the same result holds. This proves:

Theorem (Towards cohomology)

The transverse groupoid wrt the canonical transversal is a local isomorphism invariant of (P_0, R).

Table of Contents

Flow box decomposition

From now on G is unimodular.

Lemma

If $Y \subset X$ is an U^2-separated transversal, then

$$X = \bigcup_{n=1}^{\infty} X_n \quad (\textit{flow box decomposition})$$

where for each $n \in \mathbb{N}$ there exist $g_n \in G$ and $Y_n \subset Y$ such that

$$i_n : U \times Y_n \to X_n, \quad (g, y_n) \mapsto g_n g . y_n$$

is a Borel isomorphism.

Transverse measure

Fix flow box decomposition $X = \bigcup X_n$ with coordinates $i_n : U \times Y_n \to X_n$.

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- $\forall\, \mu \in \mathrm{Prob}_G(X)\ \exists!\, \nu \in M_{\mathrm{fin}}(Y)\ \forall n \in \mathbb{N}$:

$$(i_n)_*(m_G \otimes \nu)|_{U \times Y_n} = \mu|_{X_n}. \qquad (1)$$

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Notation

- $\nu = {}_G\mathrm{Res}_Y^X \mu$ (transverse measure or Palm measure)
- $\mu = {}_G\mathrm{Ind}_Y^X \nu$ (induced measure)

First examples

Example (Fully periodic case)

$X = \Gamma\backslash G \supset Y = \{\Gamma\}$, $\mu = m_{\Gamma\backslash G}$

$$\implies \nu = \frac{1}{\mathrm{covol}(\Gamma)}\delta_\Gamma, \quad \nu(Y) = \frac{1}{\mathrm{covol}(\Gamma)}.$$

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Example (Model set case)

$X = \Gamma\backslash(G \times H) \supset Y = \Gamma\backslash(\{e\} \times W) \cong W$, $\mu = m_{\Gamma\backslash(G\times H)}$

$$\implies \nu = \frac{1}{\mathrm{covol}(\Gamma)} m_H|_W, \quad \nu(Y) = \frac{m_H(W)}{\mathrm{covol}(\Gamma)}.$$

We have seen these formulas in Max' talk!

Relation to transverse point process

Proposition

Let (X, μ, Y) be a transverse tripel with transverse measure ν and transverse point process $\xi : (X, \mu) \to \mathrm{UD}(G)$, $\xi(x) = Y_x$.

1. *$\nu(Y) = i_\xi$ is the intensity of the transverse point process.*

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1. $\nu(Y) = i_\xi$ *is the intensity of the transverse point process.*
2. *The autocorrelation of* ξ *is given by*

$$\eta_\xi(f) = \int_Y \sum_{g \in Y_y} f(g) \, \mathrm{d}\nu(y).$$

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$$\eta_\xi(f) = \int_Y \sum_{g \in Y_y} f(g) \, \mathrm{d}\nu(y).$$

3. *The number variance of* ξ *is given by*

$$\mathrm{Var}(|\xi \cap B|) = \sum_{\lambda \in \Lambda} \rho_B(\lambda) \cdot \nu(Y \cap \lambda^{-1}.Y) - i_\xi^2 \cdot \int_G \rho_B \, \mathrm{d}m_G$$

where $\rho_B = \chi_B * \chi_B$.

Table of Contents

Application I: Patch frequencies ...

- $\xi : (\Omega, \mathbb{P}) \to \mathrm{UD}(G)$ ergodic uniformly discrete point process $\implies \exists\, P_0 \in \mathrm{UD}(G) : \mu_\xi(\Omega^\times_{P_0}) = 1$
- Set $(X, \mu) = (\Omega^\times_{P_0}, \mu_\xi|_{\Omega^\times_{P_0}})$ and consider $Y = Y^{\mathrm{can}}$.
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Proposition (Björklund-H.-Karasik)

If G has a sufficiently strong ergodic theorem, then $\nu([p, R])$ is the patch frequency of p. In particular, the distribution of ξ is uniquely determined by these patch frequencies.

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Example (A déjà vu)

For $Y = [\{e\}, 0]$ we have $\nu(Y) = i_\xi$ (the frequency of the trivial patch).

Application I: ... of model sets

Proposition (Björklund-H.-Karasik-Pogorzelski-Wackenhuth)

1. *The model set process* $\xi : (\Gamma\backslash(G \times H), m) \to (X, \mu_\xi)$ *with window* W *induces isomorphisms of measure spaces*

$$\begin{array}{ccc} (\Gamma\backslash(\{e\} \times W), (p_\Gamma)_*(\delta_e \otimes m_H|_W)) & \longrightarrow & (Y^{\mathrm{can}}, \nu) \\ \uparrow & \nearrow & \\ (W, m_H|_W) & & \end{array}$$

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2. *Under this isomorphism, cyclinder sets in* Y^{can} *correspond to acceptance domains in* W *(up to nullsets).*

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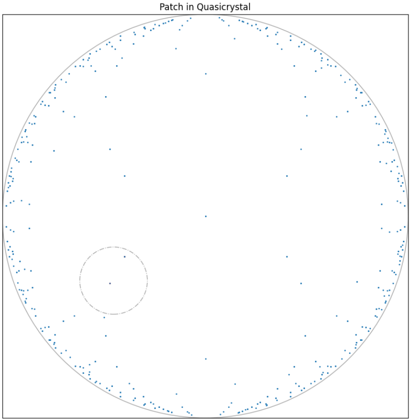
2. *Under this isomorphism, cyclinder sets in* Y^{can} *correspond to acceptance domains in* W *(up to nullsets).*
3. *In particular, if* G *has a sufficiently strong ergodic theorem, then patch frequencies are given by Haar measures of acceptance domains.*

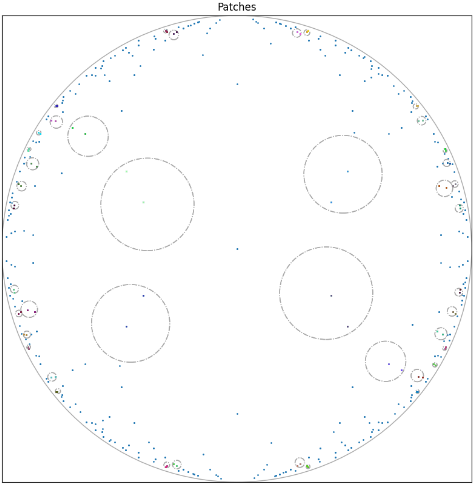
Another frequency problem



Problem

How many points in this set are centers of patches which are hyperbolic translates of the marked one?

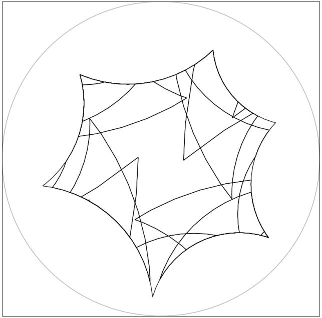
Some patches

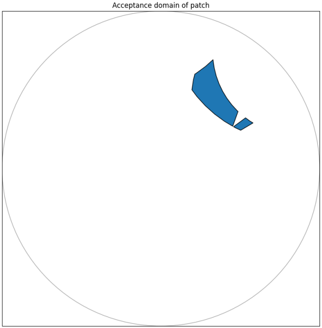


A geometric solution

Theorem (Wackenhuth)

The ratio of points which have the required patch converges, and the limit is around 5.49%. This is the hyperbolic area of the blue subset of the window divided by the total area of the window.





Application II: Intersection spaces...

Let (X_1, μ_1, Y_1), (X_2, μ_2, Y_2) be transverse ergodic G-systems with transverse measures ν_1, ν_2 and return time sets Λ_1, Λ_2

$X := G.(Y_1 \times Y_2) \subset X_1 \times X_2$ is called the intersection space, and $Y := Y_1 \times Y_2 \subset X$ is a separated transversal.

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Theorem (Björklund-H.-Karasik)

Assume that $\Lambda_1 \cap \Lambda_2$ is relatively dense in G and $\Lambda_1^3 \Lambda_2^3 \subset G$ does not accumulate at the identity. Then the intersection measure

$$\mu := {}_G\mathrm{Ind}_Y^X(\nu_1 \otimes \nu_2)$$

is finite and $\bar{\mu} := \mu(X)^{-1} \cdot \mu$ is a non-trivial joining of μ_1 and μ_2.

In particular this happens if $\Lambda_1 = \Lambda_2$ is a cocompact approximate lattice.

Application II: ... and multiple transverse recurrence

Let P_0 be a cocompact approximate lattice in G with hull X and canonical transversal Y^{can}.

Corollary (Björklund-H.-Karasik)

1. *For almost all* $(Q_1, \dots, Q_r) \in (Y^{\mathrm{can}})^r$ *there exist* $\{g_n\} \subset Q_1 \cap \cdots \cap Q_r$ *such that*

$$Q_1 g_1^{-1} \to Q_1, \quad \dots, \quad Q_r g_r^{-1} \to Q_r \quad \text{and} \quad g_n \to \infty.$$

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2. *For almost all* $(Q_1, \ldots, Q_r) \in (Y^{\mathrm{can}})^r$ *we have*

$$\underline{\mathrm{Dens}}(Q_1 \cap \cdots \cap Q_r) > 0.$$

Application III: The classical Siegel transform...

Let $H < G$ is a closed subgroup and $\Gamma < G$ is a lattice which intersects H in a lattice Γ_H.

1. The Siegel transform associated with the double fibration

$$\begin{array}{ccc} & \Gamma_H\backslash G & \\ \swarrow & & \searrow \\ H\backslash G & & \Gamma\backslash G \end{array}$$

provides a G-equivariant map $S : \mathcal{L}_c^\infty(H\backslash G) \to L^1(\Gamma\backslash G)$.

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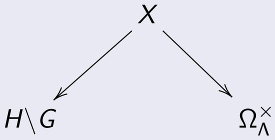
2. If $\xi \in \widehat{H}$ vanishes on Γ_H, then we also get an intertwiner from a space of test functions in $\mathrm{Ind}_H^G \xi$ to $L^1(\Gamma\backslash G)$.

Application III: ... and its aperiodic counterpart

Current project with M. Björklund

Let $H < G$ is a closed abelian subgroup and $\Lambda < G$ be an approximate lattice which intersects H in a Meyer set Λ_H. Using an intersection space



we construct for every ξ in the ε-dual of Λ_H an intertwiner from a space of test functions in $\mathrm{Ind}_H^G \xi$ to $L^1(\Omega_\Lambda^\times)$.

Application IV: Local Superrigidity

Transverse measures can also be used to simplify Machado's proof of (local) superrigidity of higher rank lattices through cocycle restriction to the transversal. But this is a story for a different talk.