Asymptotic dimension of countable approximate groups

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Suppose \((\Lambda, \Lambda^\infty)\) is a countable approximate group.

- How do you get from this countable approximate group to a metric space representing it (well enough)?
  
To rephrase:

- How do you produce the metric “nice” enough so you can make some conclusions (about the countable approximate group) regardless which “nice” metric on it you choose?

It turns out: we can define left-invariant proper metrics on the enveloping group \(\Lambda^\infty\), and two such metrics restricted to \(\Lambda\) will be coarsely equivalent.

So we will need coarse invariants, like \(\text{asdim}\), to investigate countable approximate groups with “nice” metrics.
• review \textit{dim}, introduce \textit{asdim} for metric spaces,

• list some basic properties of \textit{asdim},

• see groups as metric spaces and define their \textit{asdim},

• see countable approximate groups as metric spaces and define their \textit{asdim},

• state some recently proven theorems using \textit{asdim} for approximate groups,

• focus on one of these theorems and understand what it is saying (if time permits).

We will need to introduce some notions, like hyperbolicity for approximate groups, and more . . .
Both dimensions take their values in $\mathbb{N}_0 \cup \{\infty\}$, with $\text{dim}\emptyset := -1$.

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Definition

Let $X$ be a topological space. If $X = \emptyset$, define $\dim X := -1$. If $X \neq \emptyset$ and $n \in \mathbb{N}_0$, then $\dim X \leq n$ means: for each open cover $U$ of $X$ there is an open cover $V$ of $X$ such that

- $V$ refines $U$ (i.e., $\forall V \in V \exists U \in U$ s.t. $V \subseteq U$), and
- $\text{mult } V \leq n + 1$, i.e., any $x \in X$ lies in at most $n + 1$ elements of $V$.

We say $\dim X := n$ if $\dim X \leq n$ and $\dim X \not\leq n - 1$. If no such $n$ exists, then $\dim X := \infty$. 

\[\text{dim and asdim} \]

Definition of $\text{dim}$
Examples:

- \( \dim \) (of any discrete space) = 0
  
  (any open cover \( \mathcal{U} \) of \( X \) has \( \mathcal{V} = \{\{x\} \mid x \in X\} \) as its refinement, and \( \text{mult} \ \mathcal{V} = 1 \))

  In particular, for \( \mathbb{Z}^n \subset (\mathbb{R}^n, d_E) \), \( \dim \mathbb{Z}^n = 0 \).

- \( \dim (I^{\aleph_0}) = \infty \), where \( I^{\aleph_0} = \prod_{i=1}^{\infty} [0, 1]_i \) (Hilbert cube)

  \( (I^{\aleph_0} \text{ with metric } d((x_i), (y_i)) = \sqrt{\sum_{i \in \mathbb{N}} \frac{(d_E(x_i, y_i))^2}{i^2}} \text{ is bounded} ) \)

- \( \dim \mathbb{R}^n = n, \ \forall n \in \mathbb{N} \)

- \( \dim (n\text{-manifold}) = n \)
Definition

Let \((X, d)\) be a nonempty metric space and let \(n \in \mathbb{N}_0\).

Then \(\text{asdim } X \leq n\) means: for each uniformly bounded cover \(U\) of \(X\) there is a uniformly bounded cover \(V\) of \(X\) such that

- \(V\) coarsens \(U\) (i.e., \(U\) refines \(V\)), and
- \(\text{mult } V \leq n + 1\).

We say \(\text{asdim } X := n\) if \(\text{asdim } X \leq n\) and \(\text{asdim } X \nless n - 1\).

If no such \(n\) exists, then \(\text{asdim } X := \infty\).
Examples:

- \(\text{asdim}\) (of any bounded metric space) = 0
  (any uniformly bounded cover \(\mathcal{U}\) of \(X\) has \(\mathcal{V} = \{X\}\) as its coarsening, which is uniformly bounded because \(X\) is bounded, so \(\text{mult} \mathcal{V} = 1\))
  In particular, for Hilbert cube, \(\text{asdim} I^{\aleph_0} = 0.\) [\(\text{dim} I^{\aleph_0} = \infty\)]

- \(\text{asdim}\) (of a discrete space) can be anything.
  In particular, for \(\mathbb{Z}^n \subset (\mathbb{R}^n, d_E)\), \(\text{asdim} \mathbb{Z}^n = n.\) [\(\text{dim} \mathbb{Z}^n = 0\)]

- \(\text{asdim}\) of a discrete group that contains a copy of \(\mathbb{Z}^n, \forall n \in \mathbb{N}\)
  is = \(\infty.\)

- \(\text{asdim} \mathbb{R}^n = n, \forall n \in \mathbb{N}\)
  (we will prove that \(\text{asdim} \mathbb{R}^n = \text{asdim} \mathbb{Z}^n, \forall n \in \mathbb{N}\)).
Equivalent definition of asdim:

**Definition (Coloring definition)**

Let \((X, d)\) be a nonempty metric space and let \(n \in \mathbb{N}_0\).

Then asdim \(X \leq n \iff \forall R > 0 \ (R < \infty)\) there is a uniformly bounded cover \(\mathcal{U}\) of \(X\) such that

- \(\mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}^{(i)}\), where
- each subfamily \(\mathcal{U}^{(i)}\) is \(R\)-disjoint, i.e., \(\forall U \neq U' \in \mathcal{U}^{(i)}\) we have \(\text{dist}(U, U') \geq R\).

We refer to \(i \in \{1, 2, \ldots, n + 1\}\) as different colors.
To show $\text{asdim}(\mathbb{R}, d_E) = 1$:

- $\text{asdim} \mathbb{R} \leq 1$, because, for any $R > 0$, we can take $\mathcal{U} = \mathcal{U}^{(1)} \cup \mathcal{U}^{(2)}$ (with sets of the same color $R$-apart),
- We will show $\text{asdim} \mathbb{Z} = \text{asdim} \mathbb{R}$, and
- $\text{asdim} \mathbb{Z} \neq 0$, because otherwise, taking $R > 1$, $\mathbb{Z}$ should have a one color cover, which is uniformly bounded and consisting of sets that are $R$-apart ...
We have \( \text{asdim } \mathbb{R}^2 \leq 2 \), because, for any \( R > 0 \), we can take \( \mathcal{U} = \mathcal{U}^{(1)} \cup \mathcal{U}^{(2)} \cup \mathcal{U}^{(3)} \) (with sets of the same color \( R \)-apart).

(We still need to show \( \text{asdim } \mathbb{R}^2 \nleq 1 \), since then \( \text{asdim } \mathbb{R}^2 = 2 \).)
Theorem (Monotonicity)

If $A \subseteq X$, then $\text{asdim } A \leq \text{asdim } X$.

Theorem (Product theorem)

$\text{asdim}(X \times Y) \leq \text{asdim } X + \text{asdim } Y$.

Therefore $\text{asdim } \mathbb{R}^n \leq n \cdot \text{asdim } \mathbb{R} = n \cdot 1 = n$.

(Still would have to explain why $\text{asdim } \mathbb{R}^n \not\leq n - 1$.)

Theorem (Functions preserving asdim)

asdim is a coarse invariant, i.e., it is preserved by coarse equivalences (so, in particular, by quasi-isometries).

Once we show that $\mathbb{Z}^n \overset{QI}{\approx} \mathbb{R}^n$, they will have the same asdim.
We introduce the notions of coarse equivalence and quasi-isometry by coarsening the notion of isometry. Recall:

- A function $f : (X, d_X) \to (Y, d_Y)$ is called an \textit{isometric embedding} if it preserves distances, i.e., $\forall x, x' \in X$ we have
  \[ d_Y(f(x), f(x')) = d_X(x, x'). \]

- A function $f : (X, d_X) \to (Y, d_Y)$ is an \textit{isometry} if $f$ is an isometric embedding and it is surjective (injectivity follows from the preservation of distances, so this function is bijective).
Coarse equivalence and quasi-isometry

Definition
A function $f : (X, d_X) \rightarrow (Y, d_Y)$ is a coarse embedding if $\exists$ non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \rightarrow [0, \infty)$ s.t. $\rho_-(t) \rightarrow \infty$ when $t \rightarrow \infty$, and $\forall x, x' \in X$ we have

$$\rho_-(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_+(d_X(x, x')).$$

In particular, if both $\rho_-$ and $\rho_+$ are linear, i.e., $\exists K \geq 1, C \geq 0$ s.t.

$$\frac{1}{K} \cdot d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq K \cdot d_X(x, x') + C,$$

we say that $f$ is a quasi-isometric embedding (QI-embedding, or, more precisely, a $(K, C)$-QI-embedding).

(For $K = 1, C = 0$: $f$ is an isometric embedding.)
Coarse equivalence and quasi-isometry

**Definition**

If there exists a quasi-isometry (coarse equivalence) between spaces $X$ and $Y$, we write $X \overset{QI}{\approx} Y$ ($X \overset{CE}{\approx} Y$).

**Definition**

- If $f : X \rightarrow Y$ is a QI-embedding and $f$ is coarsely surjective, then $f$ is called a **quasi-isometry** (shortly QI). ($(K, C, D)$-QI)
- If $f : X \rightarrow Y$ is a coarse embedding and $f$ is coarsely surjective, then $f$ is called a **coarse equivalence** (shortly CE).

Properties of metric spaces which are preserved by quasi-isometries are called **QI-invariants**, and properties preserved by coarse equivalences are called **coarse invariants**.

**Properties of metric spaces which are preserved by quasi-isometries are called QI-invariants, and properties preserved by coarse equivalences are called coarse invariants.**

If there exists a quasi-isometry (coarse equivalence) between spaces $X$ and $Y$, we write $X \overset{QI}{\approx} Y$ ($X \overset{CE}{\approx} Y$).
Coarse equivalence and quasi-isometry

Example: \( \mathbb{Z} \rightarrow \mathbb{R} \) is a QI with constants \( K = 1, C = 0, D = 1 \).

\[
\begin{array}{ccccccccccccc}
\ldots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \mathbb{Z} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \mathbb{R}
\end{array}
\]

Therefore \( \mathbb{Z} \overset{QI}{\approx} \mathbb{R} \). Recall the theorem

**Theorem (Functions preserving asdim)**

*asdim* is a coarse invariant, i.e., it is preserved by coarse equivalences (in particular, by quasi-isometries).

*That is, \( X \overset{CE}{\approx} Y \Rightarrow \text{asdim } X = \text{asdim } Y \) (in particular, \( X \overset{QI}{\approx} Y \Rightarrow \text{asdim } X = \text{asdim } Y \)).*

Consequently \( \text{asdim } \mathbb{Z} = \text{asdim } \mathbb{R} \).

**Note:** a CE between geodesic metric spaces is a QI.
To introduce \( \text{asdim} \) on groups, we need a metric.

Let \( G \) finitely generated group, \( S \) a fin.gen. set of \( G \) (\( S^{-1} = S \)).

1\(^{st}\) way, on \( G \) we introduce the word metric associated to \( S \):
\[
d_S(g, h) := ||g^{-1}h||_S \text{ (length of } g^{-1}h \text{ w.r. to } S), \forall g, h \in G.
\]
- \( d_S \) is left-invariant: \( d_S(ag, ah) = d_S(g, h), \forall a, g, h \in G \),
- \( (G, d_S) \) is a discrete metric space,
- \( (G, d_S) \) is proper (closed balls are compact).

2\(^{nd}\) way, build the Cayley graph \( \Gamma_S(G) \):
- Vertices: elements of \( G \),
- Edges: \( (g, h) \in E \text{ if } h = gs, s \in S \),
- metric on \( \Gamma_S(G) \): path-length metric, i.e., \( d(a, b) = \text{length of shortest path between } a, b \). (Each edge of length 1.)
- \( (\Gamma_S(G), d) \) is a geodesic metric space.

Turns out: \( d \) on \( V(\Gamma_S(G)) \) and \( d_S \) on \( G \) coincide, and \( G \) (identified with \( V(\Gamma_S(G)) \)) is QI to \( \Gamma_S(G) \), for any finite generating set \( S \).
Cayley graph for $\Gamma\{a, b, a^{-1}, b^{-1}\}(F_2)$
Cayley graph for $\Gamma\{a, b, a^{-1}, b^{-1}\}(F_2)$, but fancier.
More on Cayley graphs

Cayley graph depends on choice of the (fin.) generating set $S$, but:

**Theorem**

If $S$ and $S'$ are both finite generating sets for $G$, then

$$(G, d_S) \overset{QI}{\approx} (\Gamma_S(G), d_S) \overset{QI}{\approx} (\Gamma_{S'}(G), d_{S'}) \overset{QI}{\approx} (G, d_{S'}).$$

**Example:** $\Gamma_{\{1,-1\}}(\mathbb{Z})$ and $\Gamma_{\{2,3,-2,-3\}}(\mathbb{Z})$. 
asdim of finitely generated groups

**Definition**

For a finitely generated group $G$, and any fin.gen. set $S$ of $G$:

$$\text{asdim } G := \text{asdim } (G, d_S) = \text{asdim } (\Gamma_S(G), d_S)$$

Note: $\text{asdim}$ is a coarse invariant (in particular, preserved by QI), so definition does not depend on the choice of fin.gen. generating set $S$.

We can also define $\text{asdim } G := \text{asdim } ([G]_c)$, where

$$[G]_c = \{(X, d_X) | (X, d_X) \overset{CE}{\approx} (G, d_S)\}.$$  

What if $G$ is not finitely generated? Then it can be:

- $G$ countable (not fin.gen.), or
- $G$ uncountable

(A finitely generated group can have a subgroup which is not finitely generated (but it will be countable).)
Metric on groups (second: for countable groups)

For $G$ countable: can define a left-invariant proper metric $d$:

**Definition**

Let $G$ be a countable group and $S \subseteq G$ be a symmetric subset. A function $w : S \cup \{e\} \to [0, \infty)$ is called a *weight function on $S$* if it is proper and satisfies $w^{-1}(0) = \{e\}$ and $w(s) = w(s^{-1})$ for all $s \in S$.

**Lemma**

Let $S$ be a symmetric generating set of a countable group $G$ and let $w : S \cup \{e\} \to [0, \infty)$ be a weight function. Then

$$\|g\|_{S,w} := \inf \left\{ \sum_{i=1}^{n} w(s_i) \mid g = s_1 \cdots s_n, \ s_i \in S \right\}$$

defines a norm on $G$, and the associated metric $d_{S,w}$ given by

$$d_{S,w}(g, h) := \|g^{-1}h\|_{S,w}$$
is left-invariant and proper.
Theorem

*If* \(d_1\) *and* \(d_2\) *are two left-invariant proper metrics on a countable group* \(G\), *then the identity* \(\text{id} : (G, d_1) \to (G, d_2)\) *is a coarse equivalence (so* \((G, d_1) \overset{CE}{\approx} (G, d_2)\)).

So the following makes sense:

**Definition**

The **coarse class** \([G]_c\) of a countable group \(G\) is the coarse equivalence class of the metric space \((G, d)\), where \(d\) is some (hence any) left-invariant proper metric on \(G\).

- Therefore, for a countable group \(G\), define:
  \[
  \text{asdim } G := \text{asdim } (G, d),
  \]
  where \(d\) is any left-invariant proper metric on \(G\). We can also define \(\text{asdim } ([G]_c) := \text{asdim } (G, d)\), so \(\text{asdim } G = \text{asdim } (G, d) = \text{asdim } ([G]_c)\).
For uncountable groups, the following theorem gives an idea how to introduce definition of \( \text{asdim} \): 

**Theorem**

If \( G \) is a countable group, then 

\[
\text{asdim} \ G = \sup \{ \text{asdim} \ H \mid H \leq G, H \text{ finitely generated} \}.
\]

For uncountable group \( G \), define 

\[
\text{asdim} \ G := \sup \{ \text{asdim} \ H \mid H \leq G, H \text{ finitely generated} \}.
\]

But what about the choice of metric? 

Some issues here, let us not go there today!
Definition (Approximate subgroup, T. Tao, 2008)

Let \((G, \cdot)\) be a group and let \(k \in \mathbb{N}\). A subset \(\Lambda\) of \(G\) is called a \textit{k-approximate subgroup} of \(G\) if:

\[(\text{AG1})\quad \Lambda = \Lambda^{-1} \quad \text{and} \quad e \in \Lambda, \quad \text{and} \]

\[(\text{AG2})\quad \exists \text{ a finite subset } F \subseteq G \text{ s.t. } \Lambda^2 \subseteq \Lambda F \quad \text{and} \quad |F| = k. \]

We say \(\Lambda\) is an \textit{approximate subgroup} if it is a \(k\)-approximate subgroup, for some \(k \in \mathbb{N}\).

Note:

- \(\Lambda^2 = \Lambda \cdot \Lambda = \{a \cdot b \mid a, b \in \Lambda\}, \quad \Lambda \cdot F = \{a \cdot f \mid a \in \Lambda, f \in F\}.\)
- If \(\Lambda\) is an approx. subgroup, then \(\Lambda^\infty := \bigcup_{k \in \mathbb{N}} \Lambda^k\) is a group \((\Lambda^\infty \leq G)\). We call \(\Lambda^\infty\) the \textit{enveloping group} of \(\Lambda\).

We call the pair \((\Lambda, \Lambda^\infty)\) an \textit{approximate group}.

We say: \((\Lambda, \Lambda^\infty)\) is finite (countable) if \(\Lambda\) is finite (countable).
Approximate groups – examples

(1) Let \((G, \cdot) = (\mathbb{Z}, +), \ n \in \mathbb{N}\) and define
\(\Lambda := \{-n, -n + 1, \ldots, -1, 0, 1, \ldots, n - 1, n\}\).
Then \(\Lambda + \Lambda = \{-2n, \ldots, 2n\} \not\subseteq \Lambda\), but for \(F = \{-n, n\}\) we get \(\Lambda + \Lambda = \Lambda + F\), i.e., \(\Lambda\) is a 2-approximate subgroup of \(\mathbb{Z}\).
Also: \(\Lambda^\infty = \mathbb{Z}\). Therefore \((\Lambda, \mathbb{Z})\) is an approximate group.

(2) (Non-example): Let \((G, \cdot) = (\mathbb{Z}, +)\) and define
\(\Lambda := \{2^i \mid i \in \mathbb{Z}\} \cup \{0\} \cup \{-2^i \mid i \in \mathbb{Z}\}\).
Then \(\Lambda + \Lambda\) contains \(2^n + 2^{n+1} = 3 \cdot 2^n, \forall n \in \mathbb{N}\), so it contains infinitely many numbers which are not in \(\Lambda\), and the “distance” of these new numbers to \(\Lambda\) goes to \(\infty\). If \(F\) is a finite set \(\subseteq \mathbb{Z}\), then the “distance” between the numbers in \(\Lambda + F\) to \(\Lambda\) is bounded. Therefore we cannot have \(\Lambda + \Lambda \subseteq \Lambda + F\), i.e., \(\Lambda\) is not an approximate subgroup of \(\mathbb{Z}\).
(3) If $G$ is a group and $H \leq G \Rightarrow$, then $H$ is also an approximate subgroup of $G \Rightarrow$ the pair $(H, H)$ is an approximate group.

(4) If $G$ is a group and $F$ is a finite symmetric subset of $G$ which contains $e \Rightarrow (F, F^\infty)$ is an approximate group.

(5) If $\Lambda$ is an approximate subgroup of a group $G$, then $\Lambda^k$ is also an approximate subgroup of $G$, so $(\Lambda^k, \Lambda^\infty)$ is an approximate group.

(6) Cartesian product of two approximate subgroups is an approximate subgroup, the image and the pre-image of an approximate subgroup (via a group homomorphism) are approximate subgroups.
(7) Let $\Lambda^\infty := BS(1, 2) = \langle a, b \mid bab^{-1} = a^2 \rangle$
(Baumslag-Solitar group of type $(1, 2)$), and define $\Lambda := \langle a \rangle \cup \{b, b^{-1}\}$. Then $\Lambda$ is symmetric, contains $e$ and generates $\Lambda^\infty$. A calculation (using $(b^{-1}ab)^2 = a$) shows that

$$\Lambda^2 \subseteq \Lambda\{e, b, b^{-1}, b^{-1}a\},$$

hence $(\Lambda, \Lambda^\infty)$ is an approximate group.

(8) If $G$ is a locally compact group and $W$ is a relatively compact (i.e. having compact closure) symmetric neighborhood of identity $e$ in $G$, then $(W, W^\infty)$ is an approximate group.
Approximate groups – examples

(9) “Cut and project” construction on an irrational lattice in $\mathbb{R}^2$: 

\[ \text{Diagram of approximate group construction on an irrational lattice.} \]
For a countable approx. group \((\Lambda, \Lambda^\infty)\), how do we define \(\text{asdim} \Lambda\)?

Recall: for a countable group \(G\):
- there are left-invariant proper metrics on \(G\), and
- if \(d_1\) and \(d_2\) are two left-invariant proper metrics on \(G\), then \((G, d_1) \overset{CE}{\approx} (G, d_2)\),
- \(\text{asdim}\) is a coarse invariant, so
- \(\text{asdim} \ G := \text{asdim} (G, d) \ (= \text{asdim} ([G]_c))\) is well-defined (for any left-invariant proper metric \(d\) on \(G\)).

Analogously, if \((\Lambda, \Lambda^\infty)\) is a countable approximate group:
- we want to associate to it the coarse (equivalence) class \([\Lambda]_c\) of (mutually coarsely equivalent) metric spaces, and
- define \(\text{asdim} \Lambda\) to be \(\text{asdim} ([\Lambda]_c)\), i.e., \(\text{asdim}\) of any metric space representing \([\Lambda]_c\).
Lemma

If $G$ is a countable group, and $\Lambda \subseteq G$ is a subset, and if we take any two left-invariant proper metrics $d$ and $d'$ on $G$, then $\text{id} : (\Lambda, d|_{\Lambda \times \Lambda}) \to (\Lambda, d'|_{\Lambda \times \Lambda})$ is a coarse equivalence.

In particular, apply this on a countable approximate group $(\Lambda, \Lambda^\infty)$, (i.e., on $\Lambda \subseteq \Lambda^\infty$): take any left-invariant proper metric $d$ on $\Lambda^\infty$, define the (canonical) coarse class of $\Lambda$:

$$[\Lambda]_c := [(\Lambda, d|_{\Lambda \times \Lambda})]_c.$$ 

Note (independence of the ambient group): If $\Lambda$ is an approximate subgroup of a countable group $G$, and if $d$ is a left-invariant proper metric on $G$, then $d|_{\Lambda^\infty \times \Lambda^\infty}$ is a left-invariant proper metric on $\Lambda^\infty$, so $[\Lambda]_c = [(\Lambda, (d|_{\Lambda^\infty \times \Lambda^\infty})|_{\Lambda \times \Lambda})]_c$ is independent of the ambient group which is used to define it.
Note: \([\Lambda]_c\) admits a representative which is a proper metric space.

Finally, for a countable approximate group \((\Lambda, \Lambda^\infty)\), define

\[
\text{asdim} \Lambda := \text{asdim} ([\Lambda]_c)
\]

\[
= \text{asdim} \text{ of any metric space representing } [\Lambda]_c.
\]

**Lemma**

If \((\Lambda, \Lambda^\infty)\) is a countable approximate group, then \(\forall k \in \mathbb{N}\), the inclusion \(\Lambda \hookrightarrow \Lambda^k\) is a coarse equivalence, so \([\Lambda]_c = [\Lambda^k]_c\).

**Corollary**

If \((\Lambda, \Lambda^\infty)\) is a countable approximate group, then

\[
\text{asdim} \Lambda \leq \text{asdim} \Lambda^\infty, \text{ and } \text{asdim} \Lambda^k = \text{asdim} \Lambda, \forall k \in \mathbb{N}.
\]
Some theorems on $\text{asdim}$ of approximate groups and the theorems which inspired them.

**Theorem (Buyalo-Lebedeva, 2007)**

*For a hyperbolic group $G$, $\text{asdim} \, G = \dim \partial G + 1$. In fact, this is true for proper, geodesic, Gromov hyperbolic, cobounded metric spaces.*

For approximate groups:

**Theorem (Cordes-Hartnick-T.)**

*For a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, $\text{asdim} \, \Lambda = \dim \partial \Lambda + 1$. In fact, this is true for proper, geodesic, Gromov hyperbolic, quasi-cobounded metric spaces.*
Some theorems on $\text{asdim}$ of approximate groups 
and the theorems which inspired them.

**Theorem (Brodskiy-Dydak-Levin-Mitra, 2008)**

Let $h : X \to Y$ be a coarsely Lipschitz map between metric spaces. Then $\text{asdim} \ X \leq \text{asdim} \ Y + \text{asdim} \ h$, where 
$\text{asdim} \ h := \sup \{ \text{asdim} \ A \mid A \subseteq X \text{ and } \text{asdim} \ (h(A)) = 0 \}$.

For approximate groups:

**Theorem (Cordes-Hartnick-T.)**

Let $(\Xi, \Xi^\infty), (\Lambda, \Lambda^\infty)$ be countable approximate groups and let $f : (\Xi, \Xi^\infty) \to (\Lambda, \Lambda^\infty)$ be a global morphism of approximate groups. Then 
$\text{asdim} \ \Xi \leq \text{asdim} \ \Lambda + \text{asdim} ([\ker (f)]_c)$. 
Now we wish to show how to generalize to approximate groups this:

**Theorem (Buyalo-Lebedeva, 2007)**

*For a hyperbolic group $G$, $\text{asdim } G = \dim \partial G + 1$. In fact, this is true for proper, geodesic, Gromov hyperbolic, cobounded metric spaces.*

We should recall and/or define:

- the notion of being *(Gromov) hyperbolic* for a (nice enough) metric space, group, approximate group,
- *(Gromov) boundary* for a (nice enough) hyperbolic space,
- *properness, coboundedness* and *quasi-coboundedness*.

**Definition**

A metric space is *proper* if all closed balls in it are compact.
**Definition**

A geodesic metric space is called *(Gromov) hyperbolic* if \( \exists \delta \geq 0 \) such that all geodesic triangles are \( \delta \)-thin, i.e., every side of a geodesic triangle is contained in \( \delta \)-nbhd of the union of the other two sides.

This is also called being \( \delta \)-hyperbolic. Let us agree that a 0-nbhd of a triangle \( = \) the triangle, so a tripod \( Y \) in a graph is 0-hyperbolic.

**Theorem**

*(Gromov) hyperbolicity is a QI invariant for geodesic metric spaces.*
Gromov hyperbolic spaces (and groups)

This definition generalizes the metric properties of classical hyperbolic geometry and of (graphs that are) trees.

- **Some examples:**
  1. hyperbolic plane $\mathbb{H}^2$ (also $\mathbb{H}^n$, $\forall n \in \mathbb{N}_{\geq 2}$),
  2. any bounded metric space,
  3. hyperbolic groups (finitely generated groups $G$ with Cayley graph $\Gamma_S(G)$ (Gromov) hyperbolic) …

For 3 in particular: Cayley graph $\Gamma_{\{a, b, a^{-1}, b^{-1}\}}(F_2)$ of the free group of rank 2:

![Cayley graph](image-url)
Gromov boundary

**Definition**

For a proper geodesic (Gromov) hyperbolic space $X$, its **(Gromov) boundary** $\partial X$ consists of points that are equivalence classes of geodesic rays in $X$, where two geodesic rays are equivalent if they fellow-travel, i.e., they are within finite Hausdorff distance from each other $(\sup_{t \in [0, \infty)} d(\gamma(t), \gamma'(t)) < \infty)$.

Elements of $\partial X$: $\gamma(\infty)$ or $\xi$. 
Gromov boundary

Metric on $\partial X$: (vague definition a visual metric on $\partial X$)

For $\xi_1, \xi_2 \in \partial X$, and some fixed $x_0 \in X$, take a geodesic ray $\gamma_1$ from $x_0$ to $\xi_1$, and a geodesic ray $\gamma_2$ from $x_0$ to $\xi_2$. These will fellow-travel for some distance $L$, before they diverge. Define $\rho(\xi_1, \xi_2) := e^{-L}$ (or $e^{-\epsilon L}$, not a metric yet). Now if $\eta_1, \eta_2 \in \partial X$, put

$$d(\eta_1, \eta_2) := \inf \left\{ \sum_{i=1}^{n} \rho(\xi_{i-1}, \xi_i) \mid \eta_1 = \xi_0, \ldots, \xi_n = \eta_2, \ n \in \mathbb{N} \right\}.$$
Topology on $\partial X$, induced by metric $d$, can also be described by nbhd bases: for $x_0 \in X$ the base point, and for any $\xi \in \partial X$, if $\gamma$ is a geodesic ray from $x_0$ to $\xi$, take an open ball centered at any point of $\gamma$, and the “shadow” of this ball on the boundary will be an open nbhd of $\xi$ in $\partial X$. 
Gromov boundary

Some properties (for $X$ proper geodesic hyperbolic):

- two visual metrics on $\partial X$ induce the same topology on $\partial X$,
- $(\partial X, d)$ is bounded, complete, compact (for $d$ any visual metric).

**Theorem**

If $(X, d_X)$, $(Y, d_Y)$ are two proper geodesic hyperbolic spaces which are quasi-isometric, then $\partial X$ and $\partial Y$ are homeomorphic.

**Some examples:**

- $\partial H^2 \approx S^1$ ($\partial H^n \approx S^{n-1}$)
- $\partial (\Gamma_{\{a,b,a^{-1},b^{-1}\}}(F_2)) \approx$ Cantor set.
A finitely generated group $G$ is called hyperbolic if for any finite generating set $S$ of $G$, the Cayley graph $\Gamma_S(G)$ is a hyperbolic metric space.

Note that we know that:

- Cayley graphs of fin.gen. groups are geodesic metric spaces (with path-length metrics, i.e., word metrics $d_S$),
- for $S$ and $S'$ finite generating sets of $G$, we have $(\Gamma_S(G), d_S) \approx (\Gamma_{S'}(G), d_{S'})$,
- hyperbolicity is a QI invariant of geodesic metric spaces
  $\Rightarrow$ hyperbolicity of finitely generated groups is well-defined.
- Cayley graphs of fin.gen. groups are proper and geodesic, so if $G$ hyperbolic, define $\partial G := \partial(\Gamma_S(G), d_S)$. 


Gromov hyperbolic spaces (and groups)

Some examples:
- Elementary hyperbolic groups:
  - finite groups \( \Rightarrow \) Cayley graph of finite diameter,
  - \( \mathbb{Z} \) and virtually cyclic groups (containing \( \mathbb{Z} \) as a finite index subgroup)
- finitely generated free groups,
- small cancellation groups,
- fundamental groups of closed surfaces with genus \( \geq 1 \),
- fundamental groups of closed, negatively curved manifolds.

Non-examples:
- \( \mathbb{Z}^2 \) \( \cong \) \((\mathbb{R}^2, d_E))\),
- any group containing \( \mathbb{Z}^2 \) as a subgroup,
- Baumslag–Solitar groups \( B(m, n) \).
A metric space \((X, d)\) is said to be **cobounded** if there is an \(r > 0\) so that for all \(x, y \in X\) there is an isometry \(f : X \rightarrow X\) so that \(d(f(x), y) < r\).

Or, equivalently, there exists a bounded subset \(A\) of \(X\) s.t. the orbit of \(A\), under the \textit{Isometry}(\(X\)) acting on \(X\), covers \(X\).
Coboundedness and quasi-coboundedness

For a metric space \((X, d)\) and for \(K \geq 1, C \geq 0, r > 0\), we say that \(X\) is \((K, C, r)\)-quasi-cobounded if for all \(x, y \in X\) there is a \((K, C, C)\)-quasi-isometry \(f : X \to X\) such that \(d(f(x), y) < r\).

\((X, d)\) is quasi-cobounded if it is \((K, C, r)\)-quasi-cobounded, for some \(K, C, r\) as above.

(Note that \(X\) is cobounded if it is \((1, 0, 0)\)-quasi-cobounded (those maps \(f\) are isometries).)
Recall that we have defined hyperbolicity for finitely generated groups. How does this translate to approximate groups?

For a group: being finitely generated

\[ \Downarrow \]

For an approximate group:

• being algebraically finitely generated
• being geometrically finitely generated

**Definition**

For \((\Lambda, \Lambda^\infty)\) we say it is algebraically finitely generated if \(\Lambda^\infty\) is a finitely generated group.
A countable approximate group \((\Lambda, \Lambda^\infty)\) is said to be \textbf{geometrically finitely generated} if \((\Lambda, d|_{\Lambda \times \Lambda})\) is \textbf{coarsely connected}, where \(d\) is a left-invariant proper metric on \(\Lambda^\infty\).

Coarsely connected = connected by “coarse paths”: \(\exists c > 0\) s.t. \(\forall x, x' \in \Lambda\), there is a \textit{c-path} from \(x\) to \(x'\), i.e., \(\exists\) a finite sequence \(x = x_0, x_1, \ldots, x_{n-1}, x_n = x'\) in \(\Lambda\) so that
\[
d(x_i, x_{i+1}) < c, \text{ for } i = 0, \ldots, n - 1.
\]

(For a countable approximate group, being geometrically finitely generated \(\Rightarrow\) being algebraically finitely generated. But not the other way around.)
Let $(\Lambda, \Lambda^\infty)$ be a countable approximate group, and let $d$ be a left-invariant proper metric on $\Lambda^\infty$. Then: $(\Lambda, d|_{\Lambda \times \Lambda})$ is coarsely connected $\iff$ there is a representative $X \in [\Lambda]_c$ which is large-scale geodesic.

Large-scale geodesic means: $\exists a > 0, b \geq 0, c > 0$ such that $\forall x, x' \in X$ there is a $c$-path between $x, x'$ of length $\leq a \cdot d(x, x') + b$.

Now, for $(\Lambda, \Lambda^\infty)$ geometrically finitely generated, we define the internal QI type of $(\Lambda, \Lambda^\infty)$:

$$[\Lambda]_{\text{int}} := \{X \in [\Lambda]_c \mid X \text{ large-scale geodesic}\},$$

Note: • $X$ large-scale geodesic $\iff$ $X \approx^{QI}$ to a geodesic metric space,
  • For $X, X' \in [\Lambda]_{\text{int}}$, we have $X \approx^{QI} X'$. 


Note that, for \((\Lambda, \Lambda^\infty)\) geometrically finitely generated:

- \([\Lambda]_{\text{int}}\) can always be represented by a proper metric \(d\) on \(\Lambda\), called *internal metric* on \(\Lambda\) (“large-scale path metric”).

- For internal metric \(d\), \((\Lambda, d)\) is proper and large-scale geodesic, so \((\Lambda, d)\) \(\approx\) to a locally finite graph \(X_\Lambda\), which we call a *generalized Cayley graph* of \((\Lambda, \Lambda^\infty)\).

- We can choose a representative \((X, d)\) of \([\Lambda]_{\text{int}} \subset [\Lambda]_c\) which is a *proper, geodesic and quasi-cobounded metric space*. We will call such a space an *apogee* for \((\Lambda, \Lambda^\infty)\).
Recall the definition for groups: A finitely generated group $G$ is hyperbolic if one (hence any) Cayley graph $\Gamma_S(G)$ of it (with respect to a finite generating set $S$) is (Gromov) hyperbolic.

Definition (Hyperbolicity for approximate groups)

A geometrically finitely generated approximate group $(\Lambda, \Lambda^\infty)$ is said to be hyperbolic if one (hence any) apogee of it is hyperbolic. Equivalently, if some (hence any) generalized Cayley graph of it is hyperbolic.

Note: For a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, an apogee $(X, d) \in [\Lambda]_{\text{int}} \subseteq [\Lambda]_c$ is a proper geodesic hyperbolic quasi-cobounded space.
B.-L. Theorem for hyperbolic approximate groups

Theorem (Cordes-Hartnick-T.)

For a hyperbolic approximate group \((\Lambda, \Lambda^\infty)\),
\[ \text{asdim} \, \Lambda = \dim \partial \Lambda + 1. \]

In fact, this is true for proper geodesic hyperbolic quasi-cobounded metric spaces.

How do we define the (Gromov) boundary \(\partial \Lambda\):

- take any apogee \((X, d) \in [\Lambda]_{\text{int}} \subseteq [\Lambda]_c\),
- recall that, if \((X, d_X), (Y, d_Y)\) are proper geodesic hyperbolic spaces s.t. \(X \approx^{QI} Y\), then \(\partial X \approx \partial Y\),
- therefore define \(\partial \Lambda := [\partial X]_{\text{homeo}} = \) the homeomorphism class of \(\partial X\), for any apogee \((X, d) \in [\Lambda]_{\text{int}}\),
- recall that \(\dim\) is a topological invariant (i.e., preserved by homeomorphisms).
Equivalently, the first part of this theorem is saying:

**Theorem**

For any apogee $X$ of a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, we have

$$\text{asdim } X = \dim \partial X + 1.$$ 

In full generality, the theorem we prove is:

**Theorem**

For a metric space $X$ which is proper, geodesic, hyperbolic and quasi-cobounded, we have

$$\text{asdim } X = \ell\text{-dim} (\partial X, d) + 1 = \dim \partial X + 1,$$

where $d$ is any visual metric on $\partial X$.

Here $\ell\text{-dim}$ denotes *linearly controlled metric dimension*.
Outline of the proof

We need to show:

- \( \asdim X \geq \dim \partial X + 1 \) and
- \( \asdim X \leq \dim \partial X + 1 \).

The first of these two inequalities works without the assumption of coboundedness or quasi-coboundedness:

Theorem (Buyalo-Schroeder)

*If \( X \) is a proper, geodesic, hyperbolic metric space, then*

\[
\asdim X \geq \dim \partial X + 1.
\]

This is not too hard to prove, using a hyperbolic cone of \( \partial X \) and its embedding into \( X \), and then some properties of \( \dim \) ... 

Note that equality holds when \( X \) is a bounded metric space, since \( \asdim X = 0 \) and \( \dim \partial X = \dim \emptyset = -1 \).

But if \( X \) is unbounded, “\( \leq \)” does not work with only the assumptions from B.-S. Theorem, as shown in the following example:
Outline of the proof

Example (hyperbolic shish-kebab (or shashlik or skewer)): Let \( n \geq 2 \), \( \gamma : [0, \infty) \to \mathbb{H}^n \) be a geodesic ray, and let \( x_1, x_2, \ldots \) be points on \( \gamma([0, \infty)) \) such that \( d(x_k, x_{k+1}) \geq 2^{k+2}, \forall k \in \mathbb{N} \). Define

\[
X = \gamma([0, \infty)) \cup \bigcup_{k \in \mathbb{N}} B(x_k, 2^k) \subset \mathbb{H}^n.
\]

With path-length metric, \( X \) is a proper geodesic hyperbolic space, which contains arbitrarily large balls of \( \mathbb{H}^n \), so \( \text{asdim } X = n \).

But \( X \) contains a single geodesic ray, so \( \partial X \) is just one point \( \Rightarrow \text{dim } \partial X = 0 \).

So \( \text{asdim } X \not\leq \text{dim } \partial X + 1 \), since \( n \not\leq 0 + 1 \).
Outline of the proof

Let us list the main steps of the proof for

$$\text{asdim } X \leq \ell \cdot \dim (\partial X, d) + 1 \leq \dim \partial X + 1,$$

when $X$ is an unbounded, proper, geodesic, hyperbolic and quasi-cobounded space, and $d$ is any visual metric on $\partial X$.

First of all, the following lemmas are true [Cordes-Hartnick-T.]:

**L1:** $X$ is a visual space (has coarse version of the geodesic extension property)

**L2:** $(\partial X, d)$ is locally quasi-similar to itself (i.e., there are constants $\lambda \geq 1$, $K \geq 1$, and $R_0 > 1$ s.t. $\forall R > R_0$ and $\forall C \subset \partial X$ with $\text{diam } C \leq \frac{1}{R}$, $\exists$ a map $f : C \to \partial X$ such that $\forall x_1, x_2 \in C$

$$\frac{1}{\lambda} R^K (d(x_1, x_2))^K \leq d(f(x_1), f(x_2)) \leq \lambda^{\frac{K}{R^k}} \sqrt{d(x_1, x_2)}.$$

**L3:** $(\partial X, d)$ is doubling, i.e., $\exists N \in \mathbb{N}$ s.t. for all $t > 0$ and all $\xi \in \partial X$ there exist $\xi_1, \ldots, \xi_N \in \partial X$ s.t. $B(\xi, 2t) \subset \bigcup_{i=1}^{N} B(\xi_i, t)$. 
Outline of the proof

Now we use the following:

**Thm1**: [Buyalo-Schroeder] Since \((\partial X, d)\) is doubling (at small scales), then \(\ell\)-dim\((\partial X, d)\) < \(\infty\).

**Thm2**: [Buyalo-Schroeder] Any visual hyperbolic space \(X\) with \(\ell\)-dim\((\partial X, d) = n\) can be QI-embedded into the product of \(n + 1\) simplicial trees, i.e., \(\exists X \xrightarrow{QI} T_1 \times \ldots \times T_{n+1}\).

**Cor**: We know that asdim \(T_i \leq 1\), so asdim \((T_1 \times \ldots \times T_{n+1}) \leq n + 1\), by the product theorem for asdim. Therefore asdim \(X \leq n + 1 = \ell\)-dim \((\partial X, d) + 1\).

**Thm3**  [C.-H.-T.] If a metric space \((\partial X, d)\) is locally quasi-similar to itself, and \(\ell\)-dim\((\partial X, d) < \infty\), then \(\ell\)-dim\((\partial X, d) \leq \dim \partial X\).

**Prop**: In general, for a metric space \((Z, d)\): \(\ell\)-dim\((Z, d) \geq \dim Z\).

So

\[\text{asdim } X \leq \ell\text{-dim } (\partial X, d) + 1 = \dim \partial X + 1.\]
Theorem (Cordes-Hartnick-T.)

For a proper, geodesic, (Gromov) hyperbolic, quasi-cobounded metric space $X$, $\text{asdim } X = \dim \partial X + 1$.

In particular, this is true for a hyperbolic approximate group $(\Lambda, \Lambda^\infty)$, i.e., $\text{asdim } \Lambda = \dim \partial \Lambda + 1$.

- this theorem for hyperbolic approximate groups is useful in proving some interesting facts, like the fact that every non-elementary hyperbolic approximate group of $\text{asdim } = 1$ is QI to a fin. generated, non-abelian free group.

- this is then used to show that non-elementary hyperbolic approximate groups have exponential internal growth.
Matthew Cordes, Tobias Hartnick, and Vera Tonić. Foundations of geometric approximate group theory. 

Thank you!