Introduction by the organizers

Aperiodic order and approximate groups, II

August 19th, 2024

Why this talk?

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However, at least in the minds of the organizers, all of the different topics are connected. By sketching some of these connections in the beginning, we hope to convince you that mutual exchange would be beneficial.

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A problem in classical harmonic analysis

- G locally compact abelian (LCA) group, $\widehat{G} = \mathrm{Hom}_c(G, \mathbb{T})$

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Definition

Λ is harmonious if its ε-duals are syndetic, i.e.

$$\forall\, \varepsilon > 0\ \exists\, K_\varepsilon \Subset \widehat{G} : \ \widehat{G} = K_\varepsilon + \Lambda^\varepsilon$$

Problem: What is the structure of harmonious sets?

Periodic harmonious sets

Example

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- $\Gamma^\perp$ is a lattice in $\mathbb{R}^n$ (dual lattice) $\implies$ Γ harmonious.

Problem: Construct aperiodic harmonious sets!

Cut-and-project schemes and model sets

Construction (Meyer, 1970)

A cut-and-project scheme (G, H, Γ) consists of

- LCA groups G, H;
- a lattice $\Gamma < G \times H$ such that $\mathrm{pr}_G|_\Gamma$ injective, $\mathrm{pr}_H(\Gamma) \subset H$ dense.

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If moreover $W \subset H$ is relatively compact with $\mathrm{Int}_H(W) \neq \emptyset$, then

$$\Lambda(G, H, \Gamma, W) := \mathrm{pr}_G(\Gamma \cap (G \times W))$$

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Meyer embedding theorem (1970)

If $\Lambda \subset G$ is syndetic (i.e. $G = K + \Lambda$ with $K \Subset G$), then

$$\Lambda \text{ harmonious} \iff \exists \text{ model set } \Lambda(G, H, \Gamma, W) : \ \Lambda \subset \Lambda(G, H, \Gamma, W).$$

Algebraic characterizations of harmonious sets

Using the embedding theorem one can show:

Theorem (Meyer (1970), Lagarias (1995))

For G LCA and $\Lambda \subset G$ syndetic TFAE:

1. *Λ is harmonious (i.e. a subset of a model set).*
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2. $\Lambda - \Lambda$ *is uniformly discrete.*
3. $\pm\Lambda \pm \cdots \pm \Lambda$ *is uniformly discrete.*
4. Λ *is locally finite and*

$$\Lambda - \Lambda \subset \Lambda + F \text{ for some finite } F \subset G.$$

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Uniform approximate lattices

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G locally compact (LC) group, $\Lambda \subset G$.

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Corollary

Let G be abelian and $\Lambda \subset G$ be symmetric containing e_G. Then

$$\Lambda \text{ harmonious and syndetic} \iff \Lambda \text{ uniform approximate lattice.}$$

Model sets and locally compact models

Uniform model sets $\Lambda(G, H, \Gamma, W) \subset G$ can be defined from cocompact lattices in non-abelian LCA groups; their syndetic subsets are uniform approximate lattices [Björklund-H.-Pogorzelski18].

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Let $\Lambda \subset \Lambda(G, H, \Gamma, W)$ be syndetic and define the $*$-map

$$f : \langle \Lambda \rangle \hookrightarrow \mathrm{pr}_G(\Gamma) \overset{\mathrm{pr}_G}{\underset{\rightarrow}{\leftarrow}} \Gamma \xrightarrow{\mathrm{pr}_H} \mathrm{pr}_H(\Gamma) \hookrightarrow H.$$

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Then (H, f) is a locally compact model [Hrushovski] for Λ, i.e.

- $f(\Lambda) \Subset H$.
- $\exists\, U \subset H$ open: $f^{-1}(U) \subset \Lambda$.

A reformulation of Meyer's embedding theorem

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Every uniform approximate lattice in an LCA group admits a locally compact model.

This fails for uniform approximate lattices in $\mathrm{GL}_2(\mathbb{R})$ [Hrushovski20], but it is true for many classes of LC groups [Machado18-23, Hrushovski20]. The failure is related to quasimorphisms.

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Motivation: A counting problem

$$\Lambda \subset \mathbb{R}^n \text{ harmonious } \implies \Lambda \text{ uniformly discrete}$$
$$\implies \forall R > 0: \ N_R := |\Lambda \cap B_R| < \infty.$$

Motivation: A counting problem

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$$\implies \forall R > 0 : \; N_R := |\Lambda \cap B_R| < \infty.$$

Problem: Asymptotic behaviour of N_R as $R \to \infty$?
Idea: Ergodic counting (as in lattice theory)

Delone dynamical systems

- G LC group, d right-invariant proper metric on G, $R > r > 0$

$\rightsquigarrow \mathrm{Del}_{r,R}(G) := \{\Lambda \subset G \mid \Lambda$ r-uniformly discrete, R-relatively dense$\}$.

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- This is a compact G-space with action

$$g.\Lambda := \{\lambda g^{-1} \mid \lambda \in \Lambda\} \quad (g \in G, \Lambda \in \mathrm{Del}_{r,R}(G))$$

  and topology induced from the embedding

$$\delta : \mathrm{Del}_{r,R}(G) \hookrightarrow C_c(G)^*, \quad \delta_\Lambda(f) := \sum_{\lambda \in \Lambda} f(\lambda)$$

  (or, equivalenty, from the Chabauty–Fell topology).

Strong approximate lattices and point processes

Definition (Björklund-H.17)

A uniform approximate lattice $\Lambda \subset G$ is strong if

$$\Omega_\Lambda := \overline{\{g.\Lambda \mid g \in G\}} \subset \mathrm{Del}_{r,R}(G)$$

admits a G-invariant probability measure.

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- Uniform model sets with "regular" windows are strong [BHP18].

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- G amenable $\implies$ Every uniform approximate lattice is strong.
- Uniform model sets with "regular" windows are strong [BHP18].
- $\mu \in \mathrm{Prob}(\Omega_\Lambda)^G$ defines a "random Delone set" in G (point process).

Ergodic counting

Λ' random Delone set with ergodic distribution $\mu \in \mathrm{Prob}(\Omega_\Lambda)^G$.

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2. If Λ is μ-generic, then

$$N_R = |\Lambda \cap B_R| = \iota_\mu \cdot \mathrm{Vol}(B_R) + \Delta_R,$$

where $\Delta_R = o(\mathrm{Vol}(B_R))$ is the discrepancy.

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3. The discrepancy can be related to higher moments of Λ'.

Variance and autocorrelation

Proposition (Folklore)

Λ' random Delone set with ergodic distribution $\mu \in \mathrm{Prob}(\Omega_\Lambda)^G$.

1. *There exists a positive and positive-definite measure η^+ on G (autocorrelation measure) such that*

$$\mathrm{Var}[\Lambda' \cap B] = \int_G 1_B * 1_B \, d\eta^+ - \iota_\mu^2 \cdot \int_G 1_B * 1_B \, d\mathrm{Vol}.$$

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2. *If Λ is generic and G has a good pointwise ergodic theorem, then*

$$\int_G f \, d\eta^+ = \lim_{R\to\infty} \frac{1}{\mathrm{Vol}(B_R)} \sum_{x\in\Lambda} \sum_{y\in\Lambda\cap B_R} f(xy^{-1})$$

(sampling formula).

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Meyer diffraction formula

Theorem (Exotic Poisson summation formula)

If $\Lambda = \Lambda(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}, \Gamma, W)$ *with* W *"regular", then*

$$\lim_{R\to\infty} \frac{1}{\mathrm{Vol}(B_R)} \sum_{x\in\Lambda} \sum_{y\in\Lambda\cap B_R} f(xy^{-1}) = \frac{1}{\mathrm{covol}(\Lambda)^2} \sum_{\xi=(\xi_1,\xi_2)\in\Gamma^\perp} \widehat{f}(\xi_1)|\widehat{\chi_W}(\xi_2)|^2.$$

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For Λ as above we have $\mathrm{Prob}(\Omega_\Lambda)^G = \{\mu\}$, and if $\mathrm{diff}_\Lambda = \widehat{\eta^+}$ denotes the diffraction of μ, then the summation formula amounts to

$$\mathrm{diff}_\Lambda = \frac{1}{\mathrm{covol}(\Lambda)^2} \sum_{\xi=(\xi_1,\xi_2)\in\Gamma^\perp} |\widehat{\chi_W}(\xi_2)|^2 \cdot \delta_{\xi_1}.$$

Crystalline measures

1. η periodic, i.e. $\eta = \delta_\Gamma$ for lattice $\Gamma \implies \eta, \widehat{\eta}$ uniformly discrete
2. η uniformly discrete, $\widehat{\eta}$ locally finite $\implies \eta$ essentially periodic [Lev-Olevskii14].

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Definition ([Meyer16])

A measure η is crystalline if η and $\widehat{\eta}$ have locally finite support.

The study of crystalline measures goes back to the 1950ies (Guinand, Kahane, Mandelbrojt), but has recently seen some major breakthroughs ([Lev-Olevski15, Kurasov-Sarnak20, Olevskii-Ulanovskii20, Meyer21]).