Generalized locally compact models of approximate subgroups via topological dynamics (more details)

Krzysztof Krupiński
(joint work with Anand Pillay)

Instytut Matematyczny
Uniwersytet Wrocławski

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Basic model theory

- $\mathcal{L} = \{f_i, P_j, c_k : i \in I, j \in J, k \in K\}$: a *language* (or *signature*), i.e. the f_i's are function symbols, P_j's relational symbols, c_k's constant symbols. For example, $\mathcal{L}_{gr} = \{\cdot, e\}$ is the language of group theory.
- Using symbols of $\mathcal{L}$, $=$, variables, logical connectives, and quantifiers, one builds $\mathcal{L}$-*formulas*. For example, $\varphi(x) := (\exists y)(\neg(x \cdot y = y \cdot x))$ is an $\mathcal{L}_{gr}$-formula with the free variable x.
- An $\mathcal{L}$-*theory* is a set of $\mathcal{L}$-sentences (i.e. $\mathcal{L}$-formulas without free variables).
- An $\mathcal{L}$-*structure* is a set M together with interpretations of all symbols of the language; e.g. if f_i is a binary function symbol, then its interpretation is a function $f_i^M \colon M^2 \to M$.
- For an $\mathcal{L}$-structure M, we recursively define what it means that an $\mathcal{L}_M$-sentence φ holds in M, symbolically $M \models \varphi$ (where $\mathcal{L}_M := \mathcal{L} \cup \{c : c \in M\}$).

Basic model theory

- An $\mathcal{L}$-structure M is a *model* of a theory T (symbolically $M \models T$) if M satisfies all sentences from T. For example, if T is group theory (i.e. the set of the well-known three axioms), then $M \models T$ iff $(M, \cdot, e)$ is a group.
- A theory T is *consistent* if it has a model.
- **Compactness Theorem (Gödel)**. A theory T is consistent iff every finite $T_0 \subseteq T$ is consistent.
- Let $M \subseteq N$ be $\mathcal{L}$-structures. We say that M is an *elementary substructure* of N (in symbols $M \prec N$), if for every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $\bar{a} \in M^n$, $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$.
- Fix $A \subseteq M$. A *complete type* over A in variable x is a finitely satisfiable in M collection p of $\mathcal{L}_A$-formulas in the free variable x such that for every $\varphi(x) \in \mathcal{L}_A$ either $\varphi(x) \in p$ or $\neg\varphi(x) \in p$. (Here, p being *fin. sat. in* M means that for any $\varphi_1(x), \dots, \varphi_n(x) \in p$ there is $m \in M$ such that $M \models \varphi_1(m) \wedge \dots \wedge \varphi_n(m)$.)

Basic model theory

- For a fixed $A \subseteq M$ the collection of all types over A in the variable x is denoted by $S_x(A)$ or $S_1(A)$.
- For any $\mathcal{L}$-structure M, $A \subseteq M$, and $p \in S_1(A)$ there exists $N \succ M$ and $n \in N$ such that $n \models p$, i.e. $p = \mathrm{tp}(n/A) := \{\varphi(x) \in \mathcal{L}_A : N \models \varphi(n)\}$.
- $S_1(A)$ is a 0-dim, compact space with a basis of the topology consisting of the clopens $[\varphi(x)] := \{p \in S_1(A) : \varphi(x) \in p\}$ for $\varphi(x) \in \mathcal{L}_A$.
- $\mathrm{Def}_A(M) := \{\varphi(M) : \varphi(x) \in \mathcal{L}_A\}$, where $\varphi(M) := \{m \in M : M \models \varphi(m)\}$.
- Fact. $S_1(A) \approx S(\mathrm{Def}_A(M))$ via $p \mapsto \{\varphi(M) : \varphi(x) \in p\}$.
- For $M \prec N$, $B \subseteq N$ a type $p \in S_1(B)$ is *fin. sat. in* M if any finite collection of formulas in p has a common realization in M. Let $S_{1,M}(B)$ be the collection of all types in $S_1(B)$ fin. sat. in M.

Semigroup operation on βG

Let $(G, \cdot)$ be a group.

In the rest of the talk, $M := (G, \cdot, \text{ all subsets of } G)$.

Then $\mathrm{Def}_M(M) = \mathcal{P}(G)$ and $S_1(M) \approx \beta G$, so we identify $S_1(M)$ and βG. This space is a G-flow (even G-ambit, where $p_e := \mathrm{tp}(e/M)$ has dense orbit) under the action by left translations: working with $S_1(M)$, $g \cdot \mathrm{tp}(a/M) := \mathrm{tp}(ga/M)$.

Fact

There is a unique left continuous semigroup $*$ operation on βG extending the action of G (i.e. $p_g * p = gp$), which is given by the following formulas:

1. For $p \in \beta G$ let $d_p \colon \mathcal{P}(G) \to \mathcal{P}(G)$ be given by $d_p(U) := \{g \in G : g^{-1}U \in p\}$. Then for $p, q \in \beta G$, $U \in p * q \iff d_q(U) \in p$.
2. For $p, q \in S_1(M)$, $p * q = \mathrm{tp}(ab/M)$, where $b \models q$, $b \models p$ and $\mathrm{tp}(a/M, b)$ is fin. sat. in M (where $a, b \in N \succ M$).

Semigroup operation on βG

Remark

1. For every B with $M \subseteq B \subseteq N \succ M$ and $p \in S_1(M)$ there exists a unique $p' \supseteq p \in S_1(B)$ fin. sat. in M and s.t. $p \subseteq p'$.
2. The formula in item (2) of the fact is well-defined.

Proof.

(1) $p' = \{\varphi(x) \in \mathcal{L}_B : \varphi(N) \cap M \in p\}$.
(2) Consider (a, b) and (a', b') satisfying the requirements. We want to show that $\mathrm{tp}(ab/M) = \mathrm{tp}(a'b'/M)$. If not, then $\varphi(x) \in \mathrm{tp}(ab/M)$ and $\neg\varphi(x) \in \mathrm{tp}(a'b'/M)$ for some $\varphi(x) \in \mathcal{L}_M$. By (1), pick $a'' \models p$ with $\mathrm{tp}(a''/M, b, b')$ fin. sat. in M. By (1), $\mathrm{tp}(a''/M, b) = \mathrm{tp}(a/M, b)$ and $\mathrm{tp}(a''/M, b') = \mathrm{tp}(a'/M, b')$. So $\mathrm{tp}(a''b/M) = \mathrm{tp}(ab/M)$ and $\mathrm{tp}(a''b'/M) = \mathrm{tp}(a'b'/M)$. Hence, $\models \varphi(a''b) \wedge \neg\varphi(a''b')$. Since $\mathrm{tp}(a''/M, b, b')$ is fin. sat. in M, there is $c \in M$ such that $\models \varphi(cb) \wedge \neg\varphi(cb')$. So $q = \mathrm{tp}(b/M) \neq \mathrm{tp}(b'/M) = q$, a contr. □

Semigroup operation on βG

- $*$ extends the action of G, because $\mathrm{tp}(g/M, b)$ is trivially fin. sat. in M for any $g \in G$, and so $\mathrm{tp}(g/M) * \mathrm{tp}(b/M) = \mathrm{tp}(gb/M) = g\,\mathrm{tp}(b/M)$.
- Left continuity of $*$. Fix $q \in S_1(M)$, $b \models q$, and a basic open set $U = [\varphi(x)]$ in $S_1(M)$. We want to show that $V := \{p \in S_1(M) : p * q \in U\}$ is open. By Rem. (1), the restriction map $r \colon S_{1,M}(M, b) \to S_1(M)$ is a homeomorphism, so enough to show that $r^{-1}[V]$ is open.
  For any $\mathrm{tp}(a/M, b) \in S_{1,M}(M, b)$ we have

$$\mathrm{tp}(a/M, b) \in r^{-1}[V] \iff \mathrm{tp}(ab/M) \in U \iff \models \varphi(ab).$$

  Therefore, $r^{-1}[V] = [\varphi(xb)]$ is a basic open set in $S_{1,M}(M, b)$.
- Associativity of $*$ is an easy exercise (not needed).

The relevant locally compact left top. semigroup

- Let X be an approximate group and $G := \langle X \rangle$.
- $S_{X^n}(M) := \{p \in S_1(M) : X^n \in p\} = [X^n]$ is clopen in $S_1(M)$.
- $S_G(M) := \bigcup_n S_{X^n}(M)$ is a locally compact subflow of βG with the orbit $G \cdot \mathrm{tp}(e/M)$ dense in $S_G(M)$.

Remark

$S_G(M)$ is closed under $*$, so it is a locally compact left topological semigroup.

Proof.

Take $p, q \in S_G(M)$, so $p \in S_{X^n}(M)$ and $q \in S_{X^m}(M)$ for some m, n. Take $a, b \in N \succ M$ such that $b \models q$, $a \models p$, and $\mathrm{tp}(a/M, b)$ fin. sat. in M, so that $p * q = \mathrm{tp}(ab/M)$. Then $a \in X^n(N)$ and $b \in X^m(N)$, so $ab \in X^{n+m}(N)$, so $p * q \in S_{X^{n+m}}(M)$. □

The smallest sufficient algebra

- $\mathcal{B}_0$ — the Boolean subalgebra of $\mathcal{P}(G)$ generated by all left translates of $X, X^2, \ldots$
- $\mathcal{B}$ — the d-closure of $\mathcal{B}_0$
- $S_G(\mathcal{B}) := \{p \in S(\mathcal{B}) : X^n \in p \text{ for some } n\}$.

Fact

$(S_G(\mathcal{B}), p_e)$ is a locally compact G-ambit and left topological semigroup with respect to the operation given as in item (1) of an earlier fact, which extends the action of G.

Remark

The restriction map $r \colon S_G(M) \to S_G(\mathcal{B})$ is a G-ambit and semigroup epimorphism.

Thus, we get a model-theoretic description of $*$ on $S_G(\mathcal{B})$.

Existence of minimal left ideals

Key property of $*$

$p * q = r \wedge q \in S_{X^n}(M) \wedge r \in S_{X^m}(M) \implies p \in S_{X^{m+n}}(M).$

Proof.

By the formula for $*$, there are $a, b \in N \succ M$ with $a \models p$, $b \models q$, and $ab \models p * q$. Then $b \in X^n(N)$, $ab \in X^m(N)$, and so $a = (ab)b^{-1} \in X^{n+m}(N)$. Hence, $p * q \in S_{X^{n+m}}(M)$. □

Lemma

There exists a left ideal $\mathcal{M} \lhd S_G(M)$ such that $\mathcal{M} \cap S_X(M)$ is minimal non-empty.

Proof.

By compactness of $S_X(M)$ and Zorn's lemma, it is enough to show that for every $s \in S_X(M)$, the set $(S_G(M) * s) \cap S_X(M)$ is closed. By the key property, the last set coincides with $(S_{X^2}(M) * s) \cap S_X(M)$. So closedness follows from left continuity of $*$ and compactness of $S_{X^2}(M)$. □

Existence of minimal left ideals

Proposition

A minimal left ideal $\mathcal{M}$ in $S_G(M)$ exists, and every minimal left ideal in $S_G(M)$ is closed and intersects $S_X(M)$.

Proof.

We prove that every ideal $\mathcal{M}$ from the last lemma and of the form $S_G(M) * s_0$ for some $s_0 \in S_X(M)$ is minimal. For that, first we show that for any $b \in G(N)$ (where $N \succ M$), $X(N)b^{-1} \cap G \neq \emptyset$. Next, take any $s \in \mathcal{M}$. It remains to show that $(S_G(M) * s) \cap S_X(M) \neq \emptyset$. Take $b \models s$ in some $N \succ M$; then $b \in X^n(N)$ for some n. Since $X(N)b^{-1} \cap G \neq \emptyset$, we can extend it to an ultrafilter in βG concentrated on X^{n+1}. This ultrafilter corresponds to a unique $\mathrm{tp}(a/M, b)$ fin. sat. in M. Then $\mathrm{tp}(a/M) * \mathrm{tp}(b/M) = \mathrm{tp}(ab/M) \in S_X(M)$.
Closedness of minimal left ideals follows from the key property, left continuity of $*$, and compactness of all $S_{X^n}(M)$, $n < \omega$. □

Existence of idempotents

Proposition

Let $\mathcal{M} \lhd S_G(M)$ be a minimal left ideal. Then $J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\} \neq \emptyset$ and $\mathcal{M} = \bigcup_{u \in J(\mathcal{M})} u\mathcal{M}$.

Proof.

Consider any $p \in \mathcal{M}$. Then $p \in S_{X^n}(M)$ for some n. By minimality of $\mathcal{M}$, $P := \{q \in \mathcal{M} : qp = p\} \neq \emptyset$. Thus, by left continuity of $*$ and the key property, P is a nonempty closed subsemigroup of $\mathcal{M}$ contained in $S_{X^{2n}}(M)$, so it is compact. By Zorn's lemma, there exists a minimal closed subsemigroup K of P. Consider any $u \in K$. We will show that $u^2 = u$. Then, since $u \in P$, we get $p = up = u(up) \in u\mathcal{M}$, so we will be done. Let $Q := \{q \in K : qu = u\}$. By compactness of K and left continuity of $*$, Ku is a nonempty closed subsemigroup of K, so $Ku = K$ as K is minimal. Hence, $Q \neq \emptyset$. Since Q is a closed subsemigroup of K, we get that $Q = K$, in particular $u \in Q$. □

Ellis group

Standard proofs apply word for word to get

Proposition

For any minimal left ideal $\mathcal{M}$ of $S_G(M)$ and $u \in J(\mathcal{M})$, the set $u\mathcal{M}$ is a group (with $*$ as group operation).

Proposition

For any minimal left ideals $\mathcal{M}, \mathcal{N}$ of $S_{G,M}(N)$ and $u \in J(\mathcal{M}), v \in J(\mathcal{N})$ the groups $u\mathcal{M}$ and $v\mathcal{N}$ are isomorphic.

Therefore, the isomorphism type of all these groups $u\mathcal{M}$ (or just any of these groups separately) can be called the *Ellis group* of $S_G(M)$.

From now on, fix a minimal left ideal $\mathcal{M}$ and $u \in \mathcal{J}(\mathcal{M})$.

The τ-topology on $u\mathcal{M}$

Definition

For any $p \in S_G(M)$ and $Q \subseteq S_G(M)$ we define $p \circ Q$ as the set of all $r \in S_G(M)$ for which there are nets $(g_i)_{i \in I}$ in G and $(q_i)_{i \in I}$ in Q such that $\lim_i g_i = u$ and $\lim_i g_i q_i = r$.

Standard proofs yield

Lemma

The operator cl_τ on subsets of $u\mathcal{M}$ given by
$\mathrm{cl}_\tau(Q) := (u\mathcal{M}) \cap (u \circ Q) = u(u \circ Q)$ is a closure operator on $u\mathcal{M}$.

Definition

By the τ-*topology* we mean the topology on the Ellis group $u\mathcal{M}$ given by the closure operator cl_τ from the last lemma.

As $u \circ Q$ is closed in $S_G(M)$, the τ-topology is weaker than the one induced from $S_G(M)$.

The τ-topology on $u\mathcal{M}$

Fact

If $(a_i)_i$ is a net in $u\mathcal{M}$ converging to $a \in \overline{u\mathcal{M}}$, then $(a_i)_i$ converges to ua in the τ-topology.

Definition

A topological space P is *quasi locally compact* if every point $p \in P$ has a neighborhood U whose closure is quasi-compact.

Proposition

The Ellis group $u\mathcal{M}$ is a quasi locally compact T_1 space.

Proof.

The fact that it is T_1 is easy:
$\mathrm{cl}_\tau(\{p\}) = u(u \circ \{p\}) = \{u(up)\} = \{p\}$.

Proof — cont.

Consider any $q \in u\mathcal{M}$. Then $q \in S_{X^n}(M)$ for some n. Also, $u \in S_{X^m}(M)$ for some m (easy to show that $m = 2$). Let $P := S_{X^{n+m}}(M)^c \cap u\mathcal{M}$.

Claim 1. $S_{X^n}(M) \cap \mathrm{cl}_\tau(P) = \emptyset$.

Proof. Take any $p \in \mathrm{cl}_\tau(P)$. Then $p = \lim_i g_i p_i$ for some nets $(g_i)_i$ in G and $(p_i)_i$ in P with $\lim_i g_i = u$. So for sufficiently large i we have that $g_i \in X^m$ and $p_i \notin S_{X^{n+m}}(M)$. Hence, $p = \mathrm{tp}(ab/M)$ for some $a \in X^m(N)$ and $b \notin X^{n+m}(N)$ (for some $N \succ M$). Therefore, $p \notin S_{X^n}(M)$, as required. $\square$

Let $V := u\mathcal{M} \setminus \mathrm{cl}_\tau(P)$. By Claim 1 and the above choices, $q \in S_{X^n}(M) \cap u\mathcal{M} \subseteq V \subseteq S_{X^{n+m}}(M)$. In particular, V is a τ-open neighborhood of q.

Proof — cont.

Claim 2. $\mathrm{cl}_\tau(V) \subseteq S_{X^{2m+n},M}(N)$

Proof. Consider any $p \in \mathrm{cl}_\tau(V)$. Then $p = \lim_i g_i p_i$ for some nets $(g_i)_i$ in G and $(p_i)_i$ in V with $\lim_i g_i = u$. So $p = \mathrm{tp}(ab/N)$ for some $a \in X^m(N)$ and $b \in X^{n+m}(N)$. So $p \in S_{X^{2m+n}}(M)$. □

It remains to show that $\mathrm{cl}_\tau(V)$ is quasi-compact in the τ-topology. For that we need to show that any net $(p_i)_{i\in I}$ in $\mathrm{cl}_\tau(V)$ has a convergent subnet. By compactness of $S_{X^{2m+n}}(M)$, the net $(p_i)_{i\in I}$ has a subnet $(q_j)_{j\in J}$ convergent to some $r \in S_{X^{2m+n}}(M)$ in the usual topology on $S_{X^{2m+n}}(M)$. So, by the last fact, $\tau\text{-}\lim_j q_j = ur$, and clearly $ur \in \mathrm{cl}_\tau(V)$. □

A standard proof yields

Proposition

$u\mathcal{M}$ equipped with the τ-topology is a semitopological group, i.e. group operation is separately continuous.

Elaborating on the standard proof (i.e. in the quasi compact T_1 context) and using the last two propositions, we get

Proposition

Let $H(u\mathcal{M})$ be the intersection of the τ-closures of the τ-neigborhoods of u. Then $u\mathcal{M}/H(u\mathcal{M})$ is a locally compact (so Hausdorff) topological group.

Existence of a generalized locally compact model

Main Theorem

The function $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$ given by $f(g) := ugu/H(u\mathcal{M})$ is a generalized locally compact model of X with a certain compact error set C. More precisely, $f^{-1}[C] \subseteq X^{22}$ and there is a compact neighborhood V of the neutral element in H such that $f^{-1}[V] \subseteq X^{14}$ and $f^{-1}[VC] \subseteq X^{26}$.

Let $N \succ M$ be $|M|^+$-saturated; $\bar{G} := \langle X(N) \rangle$.

$F_n := \{x_1 y_1^{-1} \dots x_n y_n^{-1} : x_i, y_i \in \bar{G} \text{ and } x_i \equiv_M y_i \text{ for all } i \leq n\}$

$\tilde{F}_n := \{\mathrm{tp}(a/M) \in S_G(M) : a \in F_n\}$

$\tilde{F} := ((\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$

$C := \mathrm{cl}_\tau(\tilde{F}) \cup \mathrm{cl}_\tau(\tilde{F})^{-1}$ — this is our error set!

Proposition

C is compact, normal, and symmetric as a subset of $u\mathcal{M}/H(u\mathcal{M})$. Moreover, $C \subseteq (\tilde{F}_6 \cap u\mathcal{M})/H(u\mathcal{M}) \subseteq (S_{X^{12}}(M) \cap u\mathcal{M})/H(u\mathcal{M})$.