Generalized locally compact models of approximate subgroups via topological dynamics

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Approximate groups

Definition

A symmetric subset X of a group is an *approximate subgroup* if $X^2 \subseteq FX$ for some finite $F \subseteq \langle X \rangle$; if this F can be chosen of size K, then X is called *K-approximate subgroup*. It is *definable* in M if $X, X^2, X^3, \ldots$ are all definable in M and the restrictions $\cdot|_{X^n \times X^n} \colon X^n \times X^n \to X^{2n}$ are all definable in M.

Examples

1. $[-1,1]$ is a 2-approximate subgroup of $(\mathbb{R},+)$.
2. $\{r_1x_1 + \cdots + r_dx_d : |r_i| \leq N\}$ is a 2^d-approximate subgroup of $\mathbb{R}^d$, for any $x_1, \ldots, x_d \in \mathbb{R}^d$ and $N \in \mathbb{N}$.
3. Every compact neighborhood of the neutral element in any locally compact group is an approximate subgroup.

Locally compact models

Definition

A *locally compact model* of an approximate subgroup X is a group homomorphism $f : \langle X \rangle \to H$ to some locally compact group H s.t.:

1. $f[X]$ is relatively compact in H,
2. $f^{-1}[U] \subseteq X^m$ for some $m < \omega$ and $U \subseteq H$ an open neighborhood of e.

In the definable context, we additionally require *definability* of f:

3. For any $C \subseteq U \subseteq H$ where C is compact and U is open, there exists a definable Y such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$.

Fact

In the situation from the above definition, X is commensurable with $f^{-1}[U]$, that is finitely many left translates of each of these sets cover the other one. In fact, up to commensurability, X is recovered as $f^{-1}[V]$ for any compact neighborhood V of e.

Approximate groups — Hrushovski's theorem

Theorem (Hrushovski)

A pseudofinite approximate subgroup has a locally compact model with $m = 4$.

Using Yamabe's theorem:

Corollary (Hrushovski)

For a pseudofinite approximate subgroup X there is a commensurable approximate subgroup $Y \subseteq X^4$ which has a Lie model.

This paved the way for Breuillard, Green, and Tao to give a full classification of all finite approximate subgroups.

Theorem (Massicot, Wagner)

A definably amenable approximate subgroup has a definable locally compact model with $m = 4$.

Locally compact models not always exist

Example (Hrushovski, Krupiński, Pillay)

Let $f \colon \mathbb{F}_{a,b} \to \mathbb{Z}$ be the quasi-homomorphism given by

$$f(a^{n_1} b^{m_1} \dots a^{n_k} b^{m_k}) := \sum_{i=1}^{k} \mathrm{sgn}(n_i) + \mathrm{sgn}(m_i)$$

Then $X := \mathrm{graph}(f)$ is an approximate subgroup of $\mathbb{F}_{a,b} \times \mathbb{Z}$ which does not have any locally compact model.

Quasi-homomorphisms

Definition

For a map $f\colon G \to H$ from a group (or even semigroup) G to a group H:

$$\mathrm{error}_r(f) := \{f(y)^{-1}f(x)^{-1}f(xy) : x, y \in G\},$$
$$\mathrm{error}_l(f) := \{f(xy)f(y)^{-1}f(x)^{-1} : x, y \in G\}.$$

For $C \subseteq H$, we write $f\colon G \to H : C$ if $\mathrm{error}_r(f) \cup \mathrm{error}_l(f) \subseteq C$ and we say that f is a *quasi-homomorphism with an error set* C.

Context

From now on, unless said otherwise, let X be an approximate subgroup definable in a structure M, $G := \langle X \rangle$, $\bar{M} \succ M$ a monster model, $\bar{X} := X(\bar{M})$, $\bar{G} := \langle \bar{X} \rangle$.

Generalized locally compact models

Definition (Hrushovski/Krupiński and Pillay)

A *definable generalized locally compact model of* X is a quasi-homomorphism $f\colon G \to H : C$ for some symmetric, normal, compact subset C of a locally compact group H such that:

1. for every compact $V \subseteq H$ there is $i \in \mathbb{N}$ with $f^{-1}[V] \subseteq X^i$;
2. for every $i \in \mathbb{N}$, $f[X^i]$ is relatively compact in H;
3. there is $l \in \mathbb{N}$ such that for any compact $Z, Y \subseteq H$ with $C^l Y \cap C^l Z = \emptyset$ the preimages $f^{-1}[Y]$ and $f^{-1}[Z]$ can be separated by a definable set.

Fact

In the situation from the above definition, up to commensurability, X is recovered as $f^{-1}[VC]$ for any compact neighborhood V of e.

On the main results

Hrushovski proved the existence of a definable generalized locally compact model for an arbitrary definable approximate subgroup via an application of his notion of *core* applied in the context of *local logics*. Next, he used it to find a generalized Lie model and to get some classification results, e.g. for approximate lattices in $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Q}_p)$. Recently, Machado used it to describe the structure of approximate lattices in linear algebraic groups over local fields.

We construct a definable generalized locally compact model for an arbitrary definable approximate subgroup via the topological dynamics of certain locally compact flows arising as spaces of types, which may be more accessible to group theorists, specialists in topological dynamics, or combinatorists. We also prove that our definable generalized locally compact model is universal in a suitable category.

Semigroup operation on βG and on d-closed algebras

Let G be any group, and βG its Stone-Čech compactification.

Definition/Fact

For $p \in \beta G$ let $d_p \colon \mathcal{P}(G) \to \mathcal{P}(G)$ be given by $d_p(U) := \{g \in G : g^{-1}U \in p\}$. For $p, q \in \beta G$, define $p * q \in \beta G$ by $U \in p * q \iff d_q(U) \in p$. Then $*$ is a left continuous semigroup operation on βG which extends the action of G, i.e. $p_g * q = gq$ where p_g is the principal ultrafilter at $g \in G$.

Let $\mathcal{B} \subseteq \mathcal{P}(G)$ be a *Boolean G-algebra*, i.e. a Boolean algebra closed under left translations. For any $p \in S(\mathcal{B})$ define $d_p \colon \mathcal{B} \to \mathcal{P}(G)$ as above. We say that $\mathcal{B}$ is *d-closed* if $d_p[\mathcal{B}] \subseteq \mathcal{B}$.

Fact (Newelski)

A Boolean G-algebra $\mathcal{B} \subseteq \mathcal{P}(G)$ is d-closed iff there exists a left-continuous semigroup operation on $S(\mathcal{B})$ extending the action of G. Such a semigroup operation is then unique and given by the formula as above.

Important algebras in model theory

Fact (Newelski)

For every Boolean G-algebra $\mathcal{B} \subseteq \mathcal{P}(G)$ the smallest d-closed G-algebra containing $\mathcal{B}$ (which we call the *d-closure* of $\mathcal{B}$) is the Boolean subalgebra of $\mathcal{P}(G)$ generated by $\bigcup_{p \in S(\mathcal{B})} d_p[\mathcal{B}]$.

Let G be a group definable in a model M.

Remark

The Boolean algebra $\mathrm{Def}(G)$ of definable subsets of G is not in general d-closed. So we do NOT have a nice semigroup operation on $S_G(M) \cong S(\mathrm{Def}(G))$, where $S_G(M)$ is the space of complete types over M concentrated on G.

Remark

The Boolean algebra $\mathrm{ExtDef}(G)$ of externally definable subset of G is d-closed. So we have a left continuous semigroup operation on $S_{G,\mathrm{ext}}(M) := S(\mathrm{ExtDef}(G))$ extending the action of G.

Relevant Boolean algebras in our context

X is an approximate subgroup definable in M, $G := \langle X \rangle$.

1. $\mathcal{P}(G)$ — the Boolean algebra of all subsets of G
2. $\mathrm{ExtDef}(G)$ — the Boolean subalgebra of $\mathcal{P}(G)$ generated by all externally definable subsets of $X, X^2, \ldots$
3. $\mathcal{B}_0$ — the Boolean subalgebra of $\mathcal{P}(G)$ generated by all left translates of $X, X^2, \ldots$
4. $\mathcal{B}$ — the d-closure of $\mathcal{B}_0$
5. $\mathcal{A}$ — the Boolean subalgebra of $\mathcal{P}(G)$ generated by all two-sided translates of $X, X^2, \ldots$

Remark

$\mathcal{P}(G)$, $\mathrm{ExtDef}(G)$, and $\mathcal{B}$ are d-closed. So on each of the compact spaces βG, $S(\mathrm{ExtDef}(G))$, and $S(\mathcal{B})$ we have a unique left continuous semigroup operation $*$ extending the action of G.

Relevant locally compact semigroups

1. $S_{G,\mathrm{full}}(M) := \{p \in \beta G : X^n \in p \text{ for some } n\}$.
2. $S_{G,\mathrm{ext}}(M) := \{p \in S(\mathrm{ExtDef}(G)) : X^n \in p \text{ for some } n\}$.
3. $S_G(\mathcal{B}) := \{p \in S(\mathcal{B}) : X^n \in p \text{ for some } n\}$.

Remark

$(S_{G,\mathrm{full}}(M), *)$, $(S_{G,\mathrm{ext}}(M), *)$, and $(S_G(\mathcal{B}), *)$ are locally compact left topological semigroups and "G-flows", in fact "G-ambits" with the orbit of the principal ultrafilter at e being dense. Each of these semigroups is the union of the increasing clopen compact subsets consisting of the ultrafilters concentrated on X^n, $n \in \mathbb{N}$.

One can use any of these semigroups in our construction, but getting different versions of definablity (i.e. item (3) of the definition) and in consequence universality in different categories.

Model-theoretic description of $*$

$N \succ M$ — $|M|^+$-saturated extension; $D(x)$ — definable in M
$S_x(N)$ — complete types over N; $S_D(N) := \{p \in S_x(N) : D \in p\}$
$S_{x,M}(N) := \{p \in S_x(N) : p \text{ fin. sat. in } M\} \subseteq_{\text{closed}} S_x(N)$
$S_{D,M}(N) := \{p \in S_D(N) : p \text{ fin. sat. in } M\} \subseteq_{\text{closed}} S_D(N)$
$S_{G,M}(N) := \bigcup_{n<\omega} S_{X^n,M}(N) \subseteq_{\text{open}} S_{x,M}(N)$

Fact

$S_{G,M}(N) \approx S_{G,\text{ext}}(M)$, and the induced left continuous semigroup operation $*$ on $S_{G,M}(N)$ is given by $p * q := \text{tp}(ab/N)$, where $b \models q$ and $a \models p$ such that $\text{tp}(a/Nb)$ is fin. sat. in M.

Fact

Take $M := \langle X \rangle$ with the full structure (i.e. with predicates for all subsets of M). Then $S_{G,\text{full}}(M) = S_{G,\text{ext}}(M) = S_G(M)$, and the induced semigroup operation on $S_G(M)$ is given by $p * q := \text{tp}(ab/M)$, where $b \models q$ and $a \models p$ such that $\text{tp}(a/Mb)$ is fin. sat. in M.

Model-theoretic description of $*$

Remark

The restriction maps $r_1 \colon S_{G,\mathrm{full}}(M) \to S_{G,\mathrm{ext}}(M)$ and $r_2 \colon S_{G,\mathrm{ext}}(M) \to S_G(\mathcal{B})$ are G-flow and semigroup epimorphisms.

Thus, the facts from the previous page give us model-theoretic descriptions of $*$ on $S_G(\mathcal{B})$.

I will sketch the construction of generalized locally compact models using $S_{G,\mathrm{ext}}(M)$ (as in the paper). The same construction works for $S_G(\mathcal{B})$ and for $S_{G,\mathrm{full}}(M)$ (but the latter one does not yield any information about definability).

Existence of minimal left ideals

Key property of $*$

$p * q = r \wedge q \in S_{X^n,M}(N) \wedge r \in S_{X^m,M}(N) \implies p \in S_{X^{m+n},M}(N).$

Lemma

There exists a left ideal $\mathcal{M} \lhd S_{G,M}(N)$ such that $\mathcal{M} \cap S_{X,M}(N)$ is minimal non-empty.

Proof.

By compactness of $S_{X,M}(N)$ and Zorn's lemma, it is enough to show that for every $s \in S_{X,M}(N)$, the set $(S_{G,M}(N) * s) \cap S_{X,M}(N)$ is closed. By the key property, the last set coincides with $(S_{X^2,M}(N) * s) \cap S_{X,M}(N)$. So closedness follows from left continuity of $*$ and compactness of $S_{X^2,M}(N)$. □

Existence of minimal left ideals — cont.

Proposition

A minimal left ideal $\mathcal{M}$ in $S_{G,M}(N)$ exists, and every minimal left ideal in $S_{G,M}(N)$ is closed and intersects $S_{X,M}(M)$.

Idea of the proof

We prove that every ideal $\mathcal{M}$ from the last lemma and of the form $S_{G,M}(N) * s_0$ for some $s_0 \in S_{X,M}(N)$ is minimal. For that, first we show that for any $b \in \bar{G}$, $\bar{X}b^{-1} \cap G \neq \emptyset$. Next, take any $s \in \mathcal{M}$. It remains to show that $(S_{G,M}(N) * s) \cap S_{X,M}(N) \neq \emptyset$. Take $b \models s$; then $b \in \bar{X}^n$ for some n. Since $\bar{X}b^{-1} \cap G \neq \emptyset$, we can extend it to an ultrafilter of externally definable subsets of G concentrated on X^{n+1}. This ultrafilter corresponds to a unique $\mathrm{tp}(a/N, b)$ fin. sat. in M. Then $\mathrm{tp}(a/N) * \mathrm{tp}(b/N) = \mathrm{tp}(ab/N) \in S_{X,M}(N)$.
Closedness of minimal left ideals follows from the key property, left continuity of $*$, and compactness of all $S_{X^n,M}(N)$, $n < \omega$.

Ellis group

We extend Ellis theorem on compact left topological semigroups to our locally compact context, still using the key property several times.

Theorem

Any minimal left ideal $\mathcal{M} \lhd S_{G,M}(N)$ is the disjoint union of groups $u\mathcal{M}$, where u ranges over the idempotents in $\mathcal{M}$. Moreover, the isomorphism type of $u\mathcal{M}$ depends neither on $\mathcal{M}$ nor on u, and is called the *Ellis group* of $S_{G,M}(N)$.

Theorem

There is a topology on $u\mathcal{M}$ (called the *τ-topology*) which makes it a semitopological group which is quasi locally compact and T_1. Let $H(u\mathcal{M})$ be the intersection of the τ-closures of the τ-neigborhoods of u. Then $u\mathcal{M}/H(u\mathcal{M})$ is a locally compact (so Hausdorff) topological group.

$F_n := \{x_1 y_1^{-1} \dots x_n y_n^{-1} : x_i, y_i \in \bar{G} \text{ and } x_i \equiv_M y_i \text{ for all } i \leq n\}$
$\tilde{F}_n := \{\mathrm{tp}(a/N) \in S_{G,M}(N) : a \in F_n\}$
$\tilde{F} := ((\tilde{F}_7 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$
$C := \mathrm{cl}_\tau(\tilde{F}) \cup \mathrm{cl}_\tau(\tilde{F})^{-1}$ — this will be our error set!

Proposition

C is compact, normal, and symmetric as a subset of $u\mathcal{M}/H(u\mathcal{M})$. Moreover, $C \subseteq (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})$.

Idea of the proof

We show that $\tilde{F} \subseteq (\tilde{F}_8 \cap u\mathcal{M})/H(u\mathcal{M})$. On the other hand, $\tilde{F}_n \subseteq S_{X^{2n},M}(N)$ which implies that $\mathrm{cl}_\tau(\tilde{F}_n \cap u\mathcal{M})$ is quasi-compact. All of this implies that C is compact.

Main Theorem

The function $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$ given by $f(g) := ugu/H(u\mathcal{M})$ is a definable generalized locally compact model of X with the distinguished error set C from the previous slide, which is witnessed by $l = 2$. Moreover, $f^{-1}[C] \subseteq X^{30}$ and there is a compact neighborhood V of the neutral element in H such that $f^{-1}[V] \subseteq X^{14}$ and $f^{-1}[VC] \subseteq X^{34}$.

Recall that the definability of f with $l = 2$ means that for any compact $Z, Y \subseteq H$ with $C^2Y \cap C^2Z = \emptyset$ the preimages $f^{-1}[Y]$ and $f^{-1}[Z]$ can be separated by a definable set.

Why C is an error set of f?

We explain why $\mathrm{error}_r(f) \subseteq (\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M}) \subseteq C$. We need to show that $(uhu)^{-1}(ugu)^{-1}ughu \in \tilde{F}_3$ for any $g, h \in G$.

$(*)$ $u^2 = u \Longrightarrow u \in \tilde{F}_1$.
$(**)$ $(ugu)^{-1} = \mathrm{tp}(xy^{-1}g^{-1}/N)$ for some $x \equiv_N y$.
$(***)$ $(uhu)^{-1} = \mathrm{tp}(zt^{-1}h^{-1}/N)$ for some $z \equiv_N t$, and we can choose z, t so that $\mathrm{tp}(zt^{-1}h^{-1}/N, xy^{-1}g^{-1})$ is fin. sat. in M.

By $(**)$ and $(***)$, $(uhu)^{-1}(ugu)^{-1}ughu = \mathrm{tp}(\chi/N)$, where $\chi = zt^{-1}h^{-1}xy^{-1}g^{-1}gh\alpha = zt^{-1}x^h(y^h)^{-1}\alpha$ for some $\alpha \models u$. Since $z \equiv_M t$, $x^h \equiv_M y^h$, and $\alpha \in F_1$ by $(*)$, we get that $\chi \in F_3$, and so $(uhu)^{-1}(ugu)^{-1}ughu \in \tilde{F}_3$.