Twisted Siegel transforms for quasicrystalline systems

jt. w. M.B.

1. Twisted Siegel transforms for lattices
2. Transverse Systems
3. Intersection spaces
4. Twisted Siegel transforms

(5. Examples)

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Example ("The Siegel transform")

$$C_c(\mathbb{R}^n \setminus \{0\}) \xrightarrow{S} L^1(\mathcal{L}at_1(\mathbb{R}^n))$$

$$Sf(\Gamma) = \sum_{x \in \Gamma_{vis}} f(x)$$

$$\int_{\mathcal{L}at_1(\mathbb{R}^n)} Sf \, d\mu = c_n \int_{\mathbb{R}^2} f \, d\lambda$$

Abstract Siegel transform

$\Gamma, H < G$ unimodular

$\Gamma_H := \Gamma \cap H$

$$\begin{tikzcd} & \Gamma_H \backslash G \arrow[dl, "\pi_{H \backslash G}"'] \arrow[dr, "\pi_{\Gamma \backslash G}"] & \\ H \backslash G & & \Gamma \backslash G \end{tikzcd}$$

Note: Fibers of $\pi_{H \backslash G}$ $\hat{=}$ Translates of $\Gamma_H \backslash H$

Assume: $\Gamma_H < H$ lattice

$$m_{\Gamma_H \backslash G} = \int_{H \backslash G} \nu_{Hg} \, dm_{H \backslash G}(Hg)$$

Use this to relate fcts on $H\backslash G$ to fcts on $\Gamma\backslash G$

<u>General formula:</u> Let $s: H\backslash G \to G$ be a section

$$S: C_c(H\backslash G) \longrightarrow L^1(\Gamma\backslash G)$$

$$Sf(\Gamma a) = \sum_{Hg \in H\Gamma a} f(Hg)$$

<u>Twisted formula:</u> $\xi \in \hat{H}$ with $\xi|_{\Gamma_H} \equiv 1$

$$S_\xi : \mathrm{Ind}_H^G(\xi) = C_c(H\backslash G) \longrightarrow L^1(\Gamma\backslash G)$$

<u>Ex:</u> 1) Θ-transform ($G = Sl_2(\mathbb{R})$)

2) Zak-transform ($G = \mathrm{Heis}_3(\mathbb{R})$)

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$G \curvearrowright X$

lcsc ↑ Borel, standard Borel

•) $Z \subseteq X$ Borel, $x \in X$

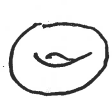
$$Z_x = \{ g \in G \mid gx \in Z \}$$ <u>hitting times</u>

$$\Lambda(Z) = \bigcup_{z \in Z} Z_z$$

•) Z <u>G-transversal</u> $\Leftrightarrow GZ = X$

$\Leftrightarrow \forall x \in X: Z_x \neq \emptyset$

$\mu \in \mathrm{Prob}(X)^G \rightsquigarrow (X, Z, \mu)$ <u>transverse system</u>

<u>Ex:</u>  $X = \Gamma\backslash\mathbb{R}^2$ torus

$\mathbb{R}^2 \curvearrowright X$, $t.[x] = [x-t]$

(1) $G = \mathbb{R}^2$, $Z = \{[0]\}$, $Z_{[x]} = \Gamma + x \subseteq \mathbb{R}^2$

$\Lambda(Z) = \Gamma$

(II) $G = \mathbb{R} \times \{0\}$. $W \subseteq \mathbb{R}$ interval

$Z = \{ [(\begin{smallmatrix} 0 \\ w \end{smallmatrix})] \mid w \in W \}$

$Z_{[\omega]} =$ ~~[illegible]~~ cut-and-project.

<u>Def:</u> (X, Z, μ) is ~~[illegible]~~

(I) <u>locally finite</u> if $|Z_x \cap K| < \infty$ $(K \subset\subset G)$ $\forall x \in X$

$\Uparrow$

(II) <u>of class L^p</u> if $x \mapsto |Z_x \cap K| \in L^p(X, \mu)$ $(K \subset\subset G)$

$\Uparrow$

(III) <u>separated</u> if $\exists U \in \mathcal{U}_G : U \cap \Lambda(Z) = \{e\}$

(IV) <u>cocompact</u> if $X = KZ$ $(K \subset\subset G)$

<u>Rem:</u> (I) (X, Z, μ) loc finite

$\Xi = \Xi(X, Z, \mu) : (X, \mu) \to LF(G)$, $x \mapsto Z_x$

<u>invariant point process</u>

(II) (X, Z, μ) separated $\Rightarrow$ Ξ unif discrete

Conversely $\Xi : (X', \mu') \to UD_r(G)$ with $\Xi \neq \emptyset$ almost surely

then $X := \mathrm{supp}(\Xi_* \mu')$

$Z := \{ Q \in X \mid e \in Q \}$

$\mu := \Xi_* \mu$

$\Rightarrow Z \subseteq X$ sep transversal

$\Xi \sim \Xi(X, Z, \mu)$

<u>Fact:</u> Every (X, μ) admits a cocompact separated transversal (Conley)

Variant: $Y \subseteq X$ H-inv transversal

$Y_x \subset {}_H\backslash G$

Say that (X, Y, μ) is H-loc finite, ...

if $Y_x \subseteq H \backslash G$ loc finite etc.

$A \times B \subseteq G \times Z$ $\quad AB = \{ gz \mid g \in A, z \in B \}$

AB flow box if $A \times B \to AB$ injective

Fact: Z separable $\Rightarrow$ $\forall x \in X$ has a flowbox nhd

$$\mu|_{AB} = m_G|_A \otimes \nu|_B$$

$\nu \in M_{fin}(Z)$ transverse measure of (X, Z, μ)

$\nu(Z)^{-1}$ covolume

Example: $X \subseteq \cup D_r(G) \setminus \{\emptyset\}$

$Z = \{ Q \in X \mid e \in X \}$

$[p, r] = \{ P \in Z \mid P \cap B_r(e) = p \}$

$\nu([p,r]) =$ patch frequency

Lemma:

$\nu \in M_{fin}(Z)$ is the unique measure s.t. $\forall F : G \times Z \to [0, \infty)$:

$$(*) \quad \int_X \sum_{g \in Z_x} F(g, gx) \, d\mu(x) = \int_G \int_Z F(g, z) \, d\nu(z) \, dm_G(g)$$

$$\int_X \underbrace{\sum_{g \in Z_x} F(g, gx)}_{TF(x)} \varphi(x) \, d\mu(x) = \int_G \int_Z F(g, z) \, \varphi(g^{-1} z) \, d\nu(z) \, dm_G(g)$$

Rem: (I) $\exists \rho : T\rho = 1$

$$\Rightarrow \int_X \varphi(x)\, d\mu(x) = \int_G \int_Z \rho(g,x)\, \varphi(g^{-1}z)\, d\nu(z)\, dm_G(g)$$

(II) ν is invariant under $\mathcal{O}_{G \ltimes X}|_Z$

Converse: if $\nu \in \mathcal{M}_{fin}(Z)$ is invariant, the ν extends to $\mu \in \mathcal{M}(X)^G$ but μ is not nec finite

$(\mu(KZ) < \infty,\ K \subset\subset G)$

Gen: (I) Z integrable (not sep) $\Rightarrow$ no flow boxes

$\Rightarrow$ (*) still works

(II) Y H-integrable $\Rightarrow$ $\sigma \in \mathcal{M}_{fin}(Y)$ similar

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BH Karasik:

$$\begin{tikzcd} & (\tilde{X}, \tilde{Z}, \tilde{\mu}) \arrow[dl] \arrow[dr] & \\ (X_1, Z_1, \mu_1) & & (X_2, Z_2, \mu_2) \end{tikzcd}$$

Construction:

•) $Y \subseteq X$ H-inv transversal

(wlog $Y = HZ$, sep transversal)

$\boxed{\text{Compatibility Condition}}$ Y is H-integrable

$\Rightarrow$ $\sigma \in \mathcal{M}_{fin}(Y)$ H-transverse measure

$$\tilde{Y} = \{H\} \times Y \subseteq {}_H\backslash^G \times X$$

$$\tilde{X} := G.\tilde{Y} \subseteq {}_H\backslash^G \times X$$

$$\tilde{\mu}(\varphi) = \int_{H\backslash G} \int_Y \varphi(Hg, g^{-1}y)\, d\sigma(y)\, dm_G(g)$$

$$
\begin{tikzcd}
 & (\tilde{X}, \tilde{Y}, \tilde{\mu}) \arrow[dl, "\substack{\text{finite}\\ \text{fibre}\\ \text{measures}}"'] \arrow[dr] & \\
(H\backslash G, \{H\}, m_{H\backslash G}) & & (X, Y, \mu)
\end{tikzcd}
$$

<u>When is Y integrable?</u>

(⊛) $\Lambda(Z) \subseteq \Lambda'$ unif approx lattice

$\Lambda' \cap H$ unif approx lattice

Z cocompact

$\Rightarrow Y \in L^\infty$ (re: getting rid of uniform & cocompact $\rightsquigarrow Y \in L^1$)

<u>Cor</u>: The hull of a strong approx lattice has an int. transversal if it intersects H largely ...

<u>Def</u> (Siegel transform)

$$S : C_c(H\backslash G) \to L^1(X, \mu)$$

$$Sf(x) = \sum_{Hg \in Y_x} f(Hg)$$

<u>Prop</u>:

(I) S is G-equiv

(II) $\int_X Sf \, d\mu = d(Y) \int_{H\backslash G} f \, dm_{H\backslash G}$

(III) $\int_X Sf(x)\, \overline{\varphi(x)}\, d\mu(x) = \int_{H\backslash G} f(Hg)\, S^* f(Hg)\, dm_{H\backslash G}(Hg)$

where $S^* : \mathcal{L}^\infty(X, \mu) \to \mathcal{L}^\infty(H\backslash G)$

$$(S^*\varphi)(Hg) = \int_Y \varphi(g^{-1}y)\, d\sigma(y)$$

3 Examples

(I) Zak transform

$$\text{Schrödinger rep} \to L^2(X_{\Lambda \leq \mathrm{Heis}})$$

(II) Θ - transform

$$^{\circ}\mathrm{Ind}_N^{SL_2(\mathbb{R})} \mathfrak{Z} \longrightarrow L^2(X_{\Lambda \leq SL_2(\mathbb{R})})$$

(III) Marklof - Strömbergsson