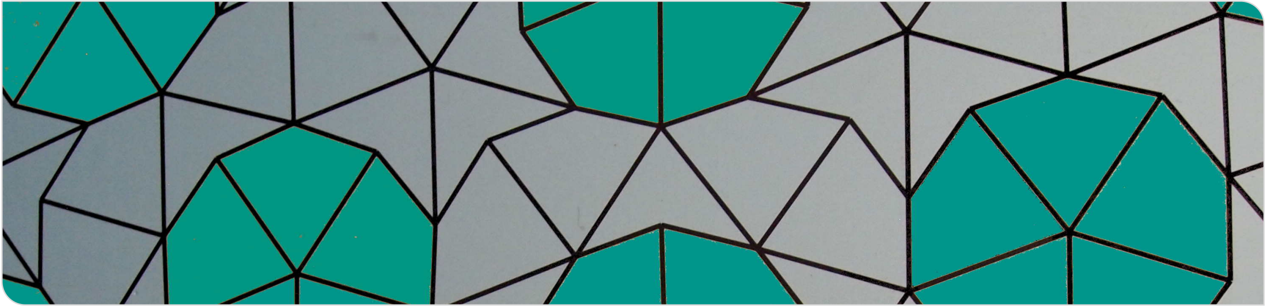
Karlsruhe Institute of Technology

Fakultät für Mathematik

Complexity of Model Sets - An Extension of the Julien-Koivusalo-Walton Approach

Worshop Aperiodic Order 2024

Peter Kaiser | August 2024



Motivation - Complexity

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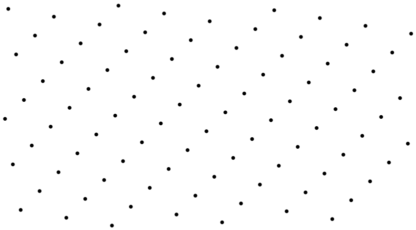

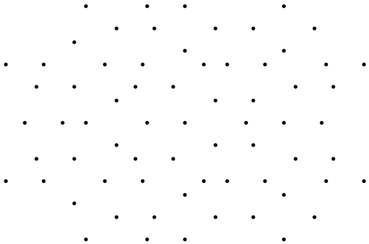

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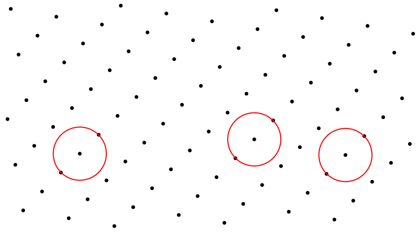

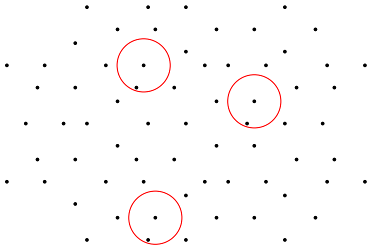

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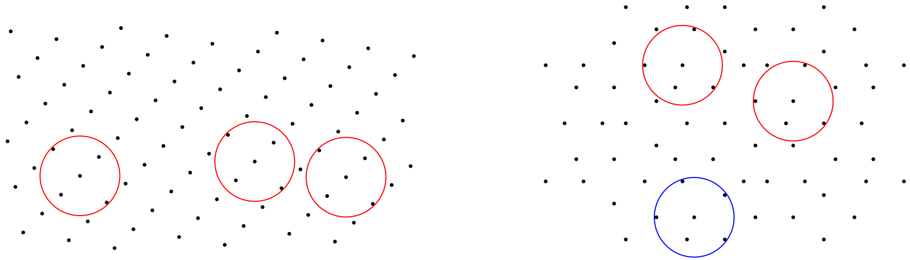

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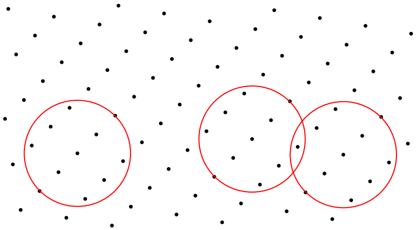
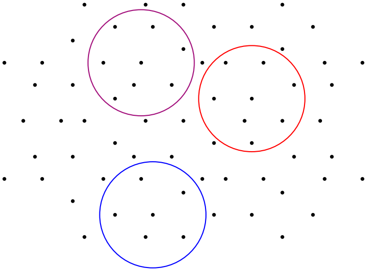
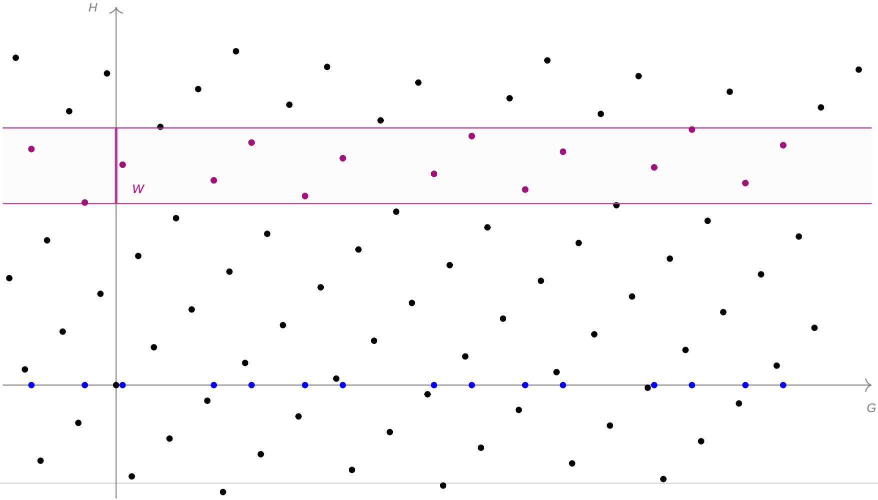




Model sets - Example 1



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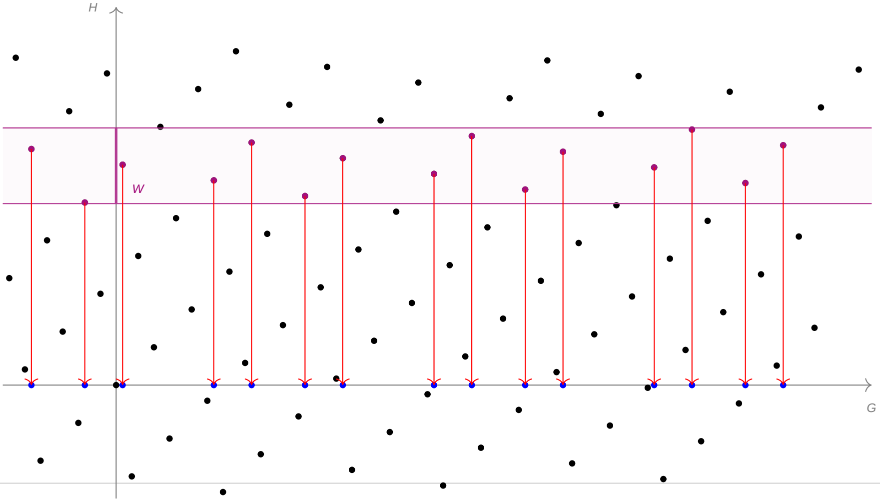
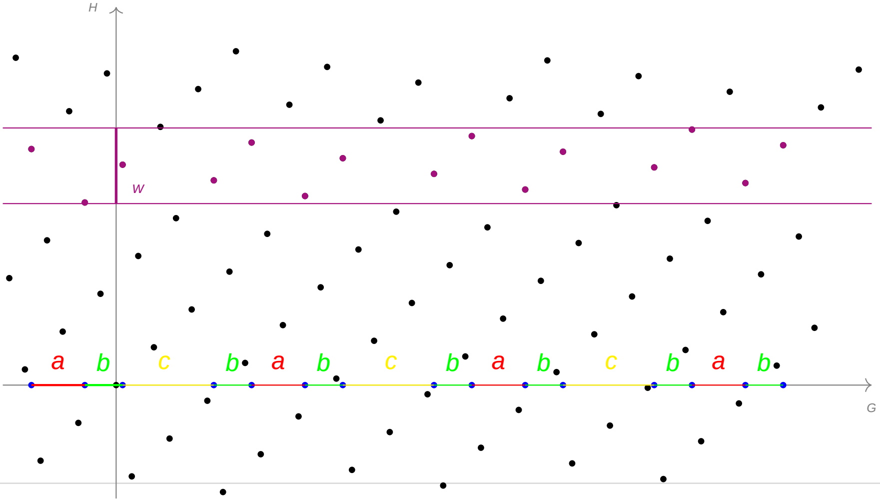


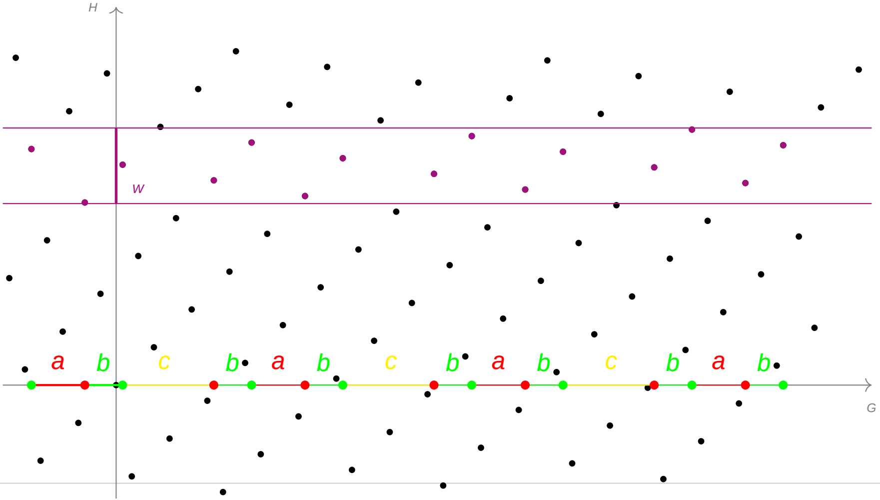
Model sets - Example 1

$$a = \sqrt{2}; b = 1; c = 1 + \sqrt{2};$$



Model sets - Example 1

$a = \sqrt{2}; b = 1; c = 1 + \sqrt{2}; r = 1.2;$



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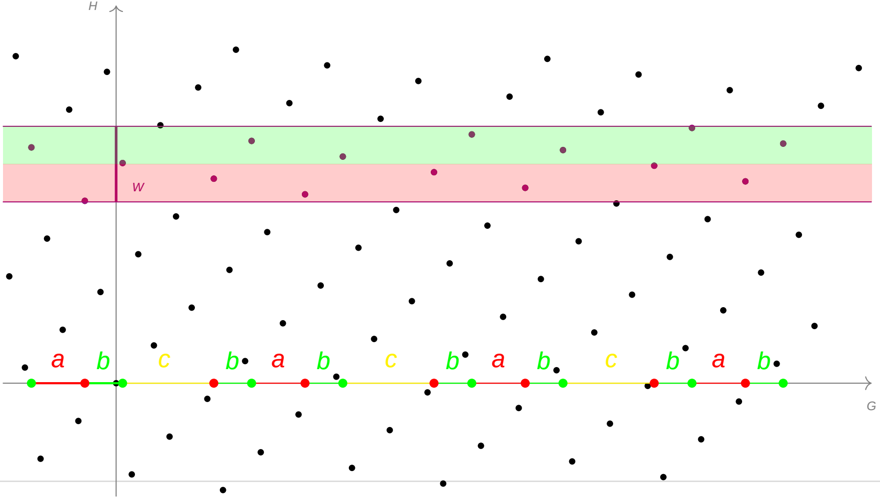
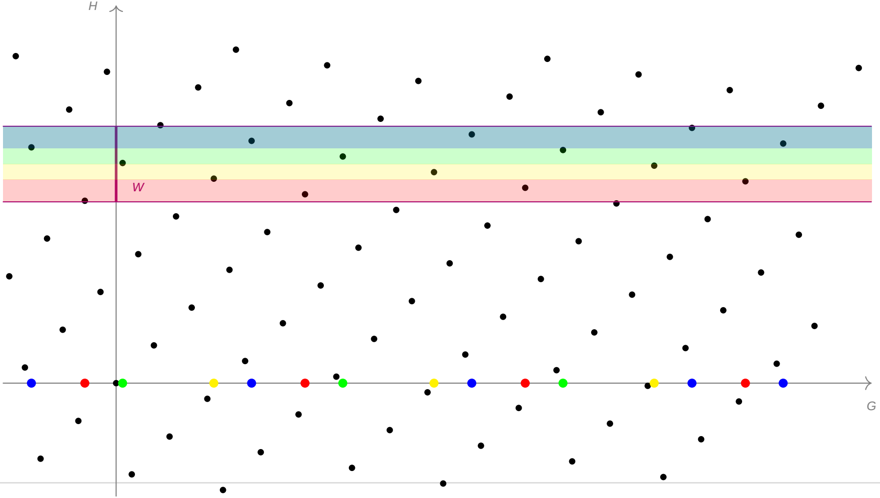


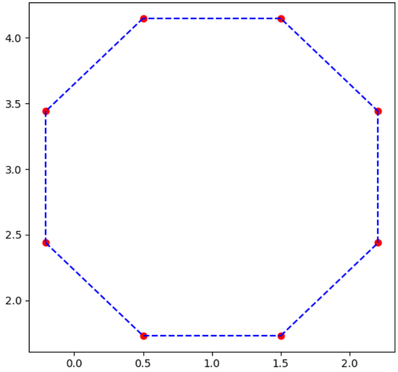
Model sets - Example 1

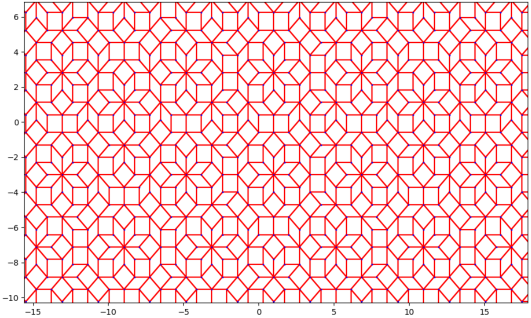
r=1.5;



Model sets - Example 2 (Ammann-Beenker)

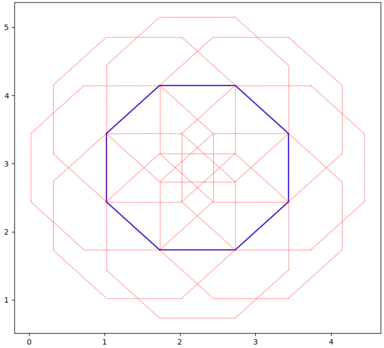
Consider $(\mathrm{R}^2; \mathrm{R}^2; \quad ; W)$;

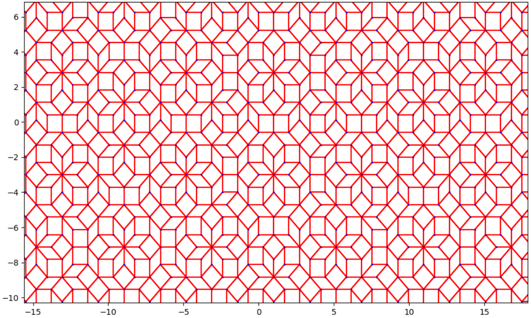




Model sets - Example 2 (Ammann-Beenker)

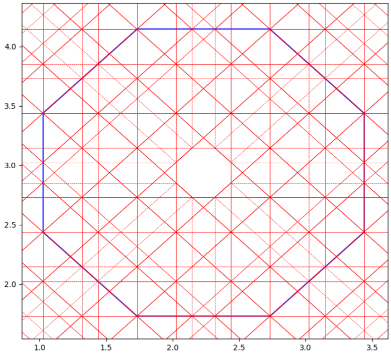
Consider $(R^2; R^2; \quad ; W)$; r=1

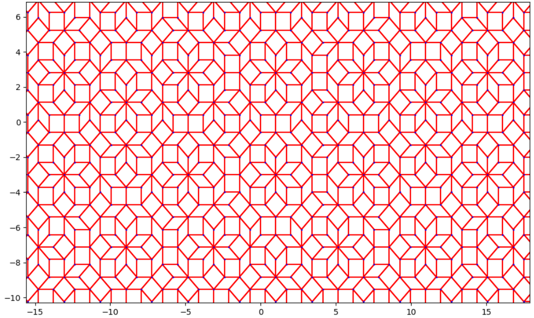




Model sets - Example 2 (Ammann-Beenker)

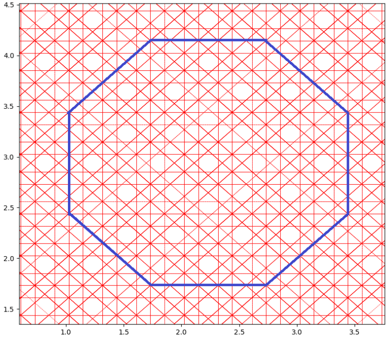
Consider $(\mathbb{R}^2; \mathbb{R}^2; \quad ; W)$; r=3

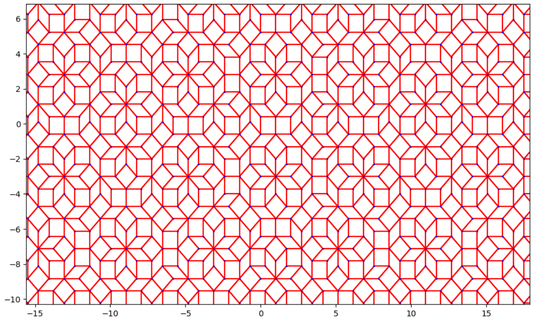




Model sets - Example 2 (Ammann-Beenker)

Consider $(R^2; R^2; \quad ; W)$; r=5





Correspondense

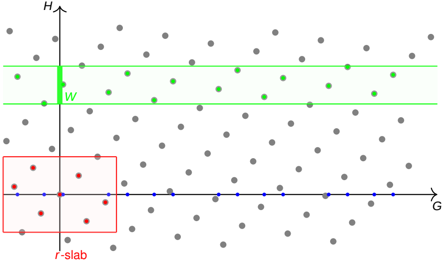
Definition

***r*-acceptance domain of**

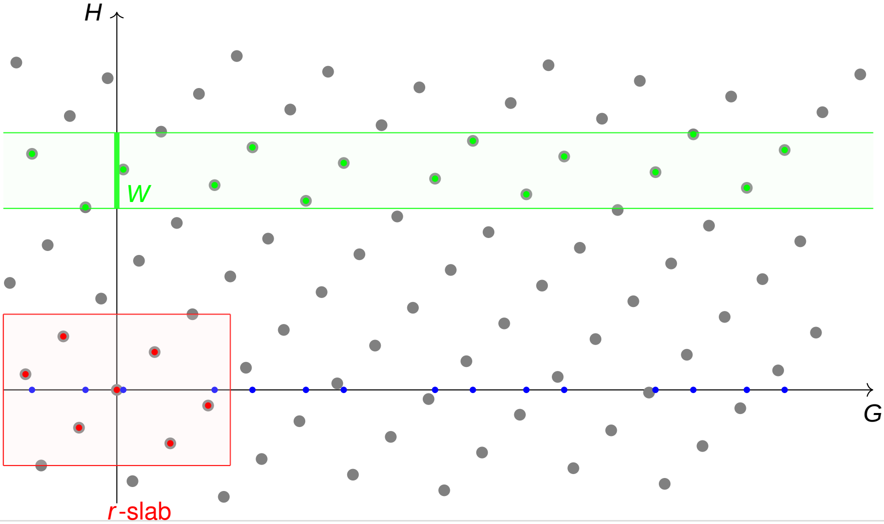
$$W_r(\;) := \left(\bigcap_{\in S_r(\;)} W \right) \cap \left(\bigcap_{' \in S_r^{C}(\;)} {}' W^{C} \right)$$

Shifts

$S_r := \{ \; _H \; (\; ; \;) \in \; | \; | < r$ and $\in WW^{-1} \; \}$;
***r*-slab**; all the possible neighbors with distance at most r (in G-direction).



r-slab



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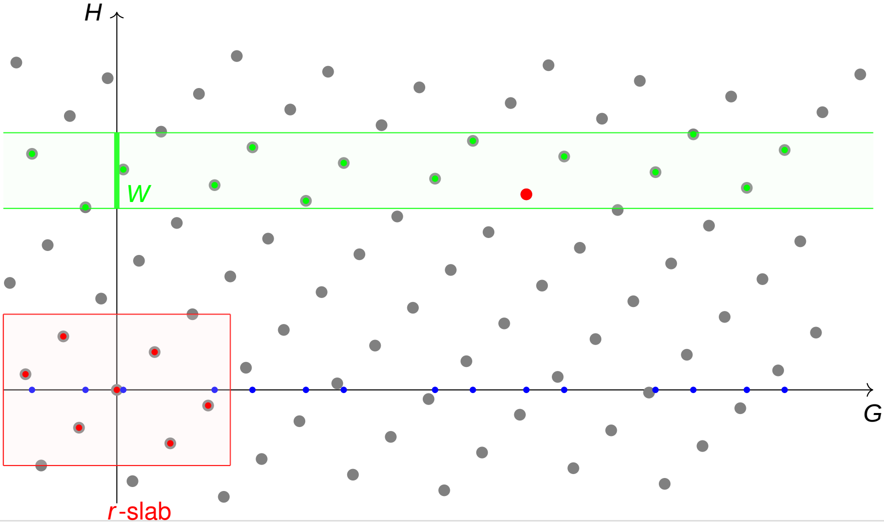

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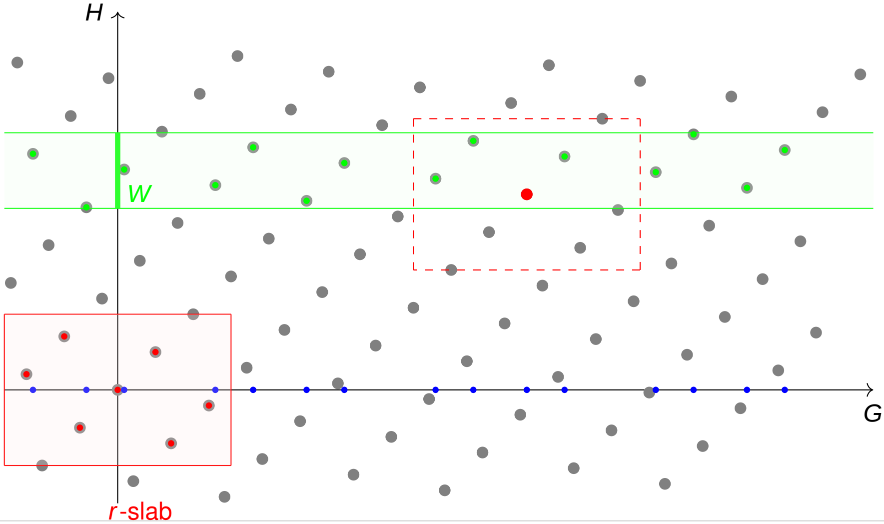

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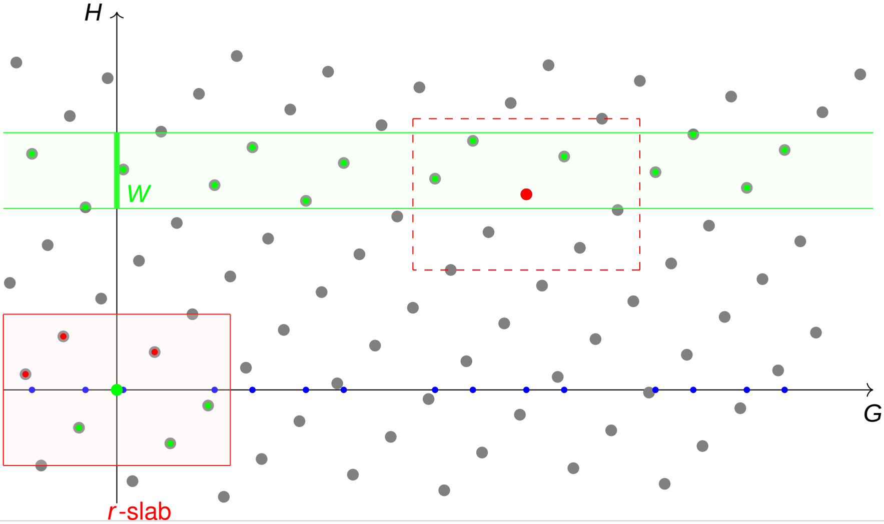

Julien-Koivusalo-Walton Approach

Theorem (Koivusalo, Walton 2021)

Let $(\mathbb{R}^{d_1}; \mathbb{R}^{d_2}; \quad ; W)$ be a polytopal model set.

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$$:= \max_{|F|=d_2} \; {}' ; \quad \textit{for} \quad {}' := \sum_{H \in F} (d_1 \quad s(H)).$$

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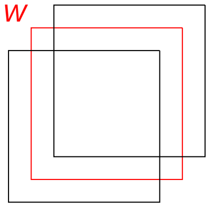
We do not consider the treatment of the stabilizers.

Julien-Koivusalo-Walton Approach - Upper bound



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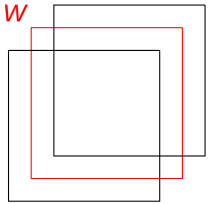
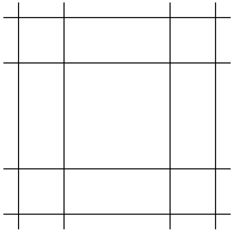


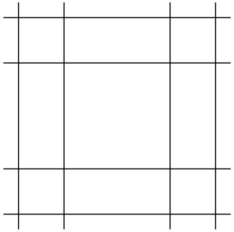
Julien-Koivusalo-Walton Approach - Upper bound

1. $p(r) \quad {}_{0} \quad W\, n \quad \mathrm{T} \quad {}_{2 S_r} \quad @W$
2. Extend the edges of the polytopes to full hyperplanes and consider R^{d_2} instead of W.



Julien-Koivusalo-Walton Approach - Upper bound

1. $p(r) \quad {}_0 \; W \setminus \bigcap_{\in S_r} \partial W$
2. Extend the edges of the polytopes to full hyperplanes and consider $\mathbb{R}^{d_2}$ instead of W.
3. $N \quad |S_r|$ hyperplanes; Zaslavsky(hyperplane arrangements) $\Rightarrow$ maximal $\sum_{i=0}^{d_2} \binom{N \; |S_r|}{i}$ connected components



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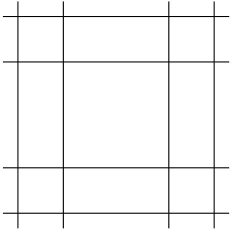

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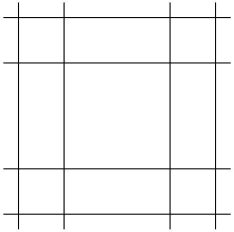
5. $|S_r| \quad (B_r^{\mathbb{R}^{d_1}})$ (point-wise ergodic theorem)



Julien-Koivusalo-Walton Approach - Lower bound

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1. $_0 \; W \, n \; {}^{\mathrm{T}}{}_{2 S_r} \; @W \quad p(r)$
2. Consider a subset of S_r and a small region B inside the window such that the faces of the window nicely cut the region.

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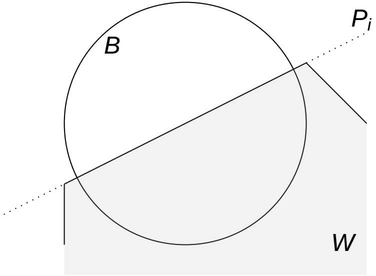

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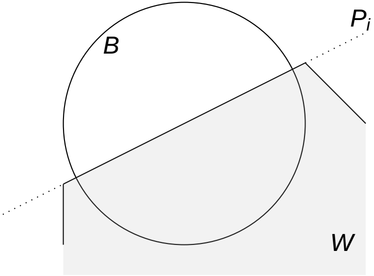

3. Consider even a smaller subset U_r S_r such that all the faces that cut B nicely intersect inside B.

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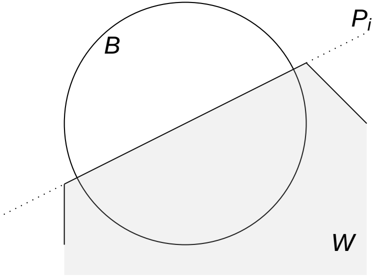

3. Consider even a smaller subset $U_r \quad S_r$ such that all the faces that cut B nicely intersect inside B.
4. Consider only d_2 many families of parallel hyperplanes. They decompose B into $jU_rj^{d_2}$ parts.

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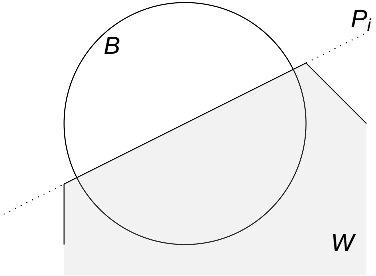

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5. $jU_rj \quad j S_r j$

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1. Point-wise Ergodic theorem for $j\, S_r\, j$,
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4. Shifts of hyperplanes stay parallel
5. Combinatorics about hyperplane arrangements

Extension Example 1

Heisenberg Group *Heis*, $(\mathbb{R}^3, \cdot)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_1 y_2 - x_2 y_1 \end{pmatrix}$$

$$d\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \max\{|x_1 - y_1|, |x_2 - y_2|, \sqrt{|x_3 - y_3 - x_1 y_2 + x_2 y_1|}\}$$

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2. Polytopal window
3. Shifts of hyperplanes stay hyperplanes
4. Shifts of hyperplanes stay parallel
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1. Point-wise Ergodic theorem (Lindenstrauss)

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1. Point-wise Ergodic theorem (Lindenstrauss)
2. Polytopes = Polytopes in $\mathbb{R}^3$

Extension Example 1

Heisenberg Group *Heis*, $(\mathbb{R}^3, \cdot)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_1 y_2 - x_2 y_1 \end{pmatrix}$$

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5. Combinatorics about hyperplane arrangements

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2. Polytopes = Polytopes in $\mathbb{R}^3$
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5. Solution: Need better combinatorics

Toppling Hyperplanes

Karlsruhe Institute of Technology

Theorem of Beck (1983)

Let H be a hyperplane arrangement in $\mathbb{R}^d$, $|H| = n$, and $B \subseteq \mathbb{R}^d$ convex,

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then $|\{p \in B \mid p = f_i \cap f_j; f_i f_j \in H; f_i \neq f_j\}| > cn^d$

Julien-Koivusalo-Walton Approach Extended 1

Theorem (K. 2022)

Let $\quad (Heis; Heis; \quad ; W)$ be a polytopal model set. And let the stabilizers of the hyperplanes that bound the window be trivial in $\quad_H$, then

$$p(r) \quad r^{4\ 3} \qquad {}_{Heis}(B_r)^{\dim(Heis)}$$

Extension Example 2

Isometry group of $\mathbb{H}^2$, $x_0 \in \mathbb{H}^2$ basepoint, $d(g, h) = \sup_{x \in \mathbb{H}} d_{\mathbb{H}}(g^{-1}.x, h^{-1}.x) e^{-d_{\mathbb{H}}(x_0, x)}$

Figure:

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2. Idea: Lifts of Polytopes in $\mathbb{H}^2$

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Hyperplanes H are Euclidean hyperplanes intersected with D^2; associated half-spaces H^+, H^-

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Lemma (Acceptance domains in lifted windows)

$\{\text{acceptance domains in } W\} \xrightarrow{1:1} \bigcap_{\gamma \in S_r(\ )} \tilde{\gamma}.W \cap \bigcap_{\gamma' \in S_r^c(\ )} \tilde{\gamma}'.W^c$

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Julien-Koivusalo-Walton Approach Extended 2

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Theorem (K. 2022)

Let $(\mathrm{Isom}(\mathrm{H}^2); \mathrm{Isom}(\mathrm{H}^2); \; ; W)$ be a polytopal model set. And let the stabilizers of the hyperplanes that bound the window be trivial in $_H$, then

$$p(r) \quad e^{2r} \quad {}_{\mathrm{Isom}(\mathrm{H}^2)}(B_r)^2$$

Julien-Koivusalo-Walton Approach Extended 3

Theorem (K. 2022)

Let $(G; H; \;; W)$ be a polytopal model set, G, H be two-step nilpotent Lie Groups. And let the stabilizers of the hyperplanes that bound the window be trivial in $_H$, then

$$p(r) \qquad {}_G(B_r)^{\dim(H)}$$

Theorem (K. 2022)

Let $(\mathrm{Isom}(\mathbb{H}^{d_1}); \mathrm{Isom}(\mathbb{H}^{d_2}); \;; W)$ be a polytopal model set. And let the stabilizers of the hyperplanes that bound the window be trivial in $_H$, then

$$p(r) \qquad {}_{\mathrm{Isom}(\mathbb{H}^{d_1})}(B_r)^{d_2}$$

Thank you for your attention.