1 Introduction

Let $G$ be a locally compact (Hausdorff) group. We call 
\[ C^{n}_{cb}(G) := \{ f \in C(G^{n+1}, \mathbb{R}) \mid f(gg_0, \ldots, gg_n) = f(g_0, \ldots, g_n), \|f\|_\infty < \infty \} \]
the space of continuous bounded homogeneous $n$-cochains and define the coboundary map 
\[ d^{n-1} : C^{n-1}_{cb}(G) \to C^n_{cb}(G) \]
by 
\[ d^{n-1}(f)(g_0, \ldots, g_n) := \sum_{j=0}^{n} (-1)^{j} f(g_0, \ldots, \hat{g}_j, \ldots, g_n). \]

This yields a cochain complex (i.e. $d^n \circ d^{n-1} = 0$), so we can consider its $n$-th cohomology 
\[ H^n_{cb}(G) := \ker(d^n) / \text{im}(d^{n-1}). \]

We call $H^n_{cb}(G)$ the $n$-th continuous bounded cohomology space of $G$ with $\mathbb{R}$-coefficients. We are interested in the continuous bounded cohomology of connected groups.

**Theorem 1.1** (Monod, 2001). Let $G$ be a locally compact second countable group and let $N$ be a closed normal amenable subgroup. Then $H^*_{cb}(G) \cong H^*_{cb}(G/N)$.

**Theorem 1.2** (Gleason-Yamabe). Let $G$ be a connected locally compact group. For every open neighborhood $U$ of the identity there exists a compact normal subgroup $K$ of $G$ in $U$ such that $G/K$ is isomorphic to a connected Lie group.

**Corollary 1.3.** Let $G$ be a connected locally compact second countable group. Then there exists a connected semisimple center-free Lie group without compact factors $G'$ such that $H^*_{cb}(G) \cong H^*_{cb}(G')$.

**Proof.** There exists a compact normal subgroup $K$ of $G$ such that $G/K$ is isomorphic to a connected Lie group. We can write 
\[ \widehat{G/K} \cong R \rtimes (S_1 \times S_2), \]
where $R$ denotes the solvable radical of $\widehat{G/K}$, $S_1$ is a connected compact semisimple Lie group and $S_2$ is a connected semisimple Lie group without compact factors. Since solvable groups and compact groups are amenable, this yields the isomorphism 
\[ H^*_{cb}(G) \cong H^*_{cb}(S_2/Z(S_2)). \]

Here, $G' = S_2/Z(S_2)$ is a connected semisimple center-free Lie group without compact factors.

**Conjecture 1.4** (Monod, 2006). Let $G$ be a connected semisimple finite-center Lie group without compact factors. Then the comparison map 
\[ H^*_{cb}(G) \to H^*_c(G), \ [c]_b \mapsto [c] \]
is an isomorphism.
Here, $H^*_c(G)$ denotes the continuous cohomology of $G$. It is known explicitly in all cases. Note that both continuous bounded cohomology and continuous cohomology are invariant under finite extensions. The comparison map is not an isomorphism for $SL_2(R)$, which has infinite center.

A Künneth formula is conjectured to hold for continuous bounded cohomology. Thus, it would suffice to check the isomorphism conjecture for connected simple center-free Lie groups without compact factors. These groups are classified.

**Degree 0/1**: Not difficult.

**Degree 2**: Proven by Burger and Monod.

**Degree 3**: Proven for $SL_n(R)$ by Burger and Monod. Proven for $SL_n(C)$ by Goncharov, Bloch, and Monod. Recently proven for the groups $SO_{2r+1}(C)$, $Sp_{2r}(C)$, and $SO_{2r}(C)$ by De la Cruz Mengual and Hartnick.

**Degree 4**: Proven for $SL_2(R)$ by Hartnick and Ott.

We will focus on the recent results by De la Cruz Mengual and Hartnick, see [1].

### 2 Bounded-Cohomological Stabilization

**Fact 2.1.** Let $G$ be a connected simple finite-center Lie group. Then

$$H^3_c(G) \neq 0 \iff G \text{ is a complex Lie group.}$$

In that case, $H^3_c(G) \cong \mathbb{R}$ is generated by the Borel class.

Consider the block embeddings

$$SL_2(C) \hookrightarrow SL_3(C) \leftrightarrow SL_4(C) \leftrightarrow \ldots$$

Monod proved that the induced sequence

$$H^3_{cb}(SL_2(C)) \hookrightarrow H^3_{cb}(SL_3(C)) \leftrightarrow H^3_{cb}(SL_4(C)) \leftrightarrow \ldots$$

consists of injections preserving the bounded Borel class. Combined with the fact that the comparison map $H^3_{cb}(SL_2(C)) \to H^3_c(SL_2(C))$ is an isomorphism mapping the bounded Borel class to the Borel class, this implies $H^3_{cb}(SL_r(C)) \cong \mathbb{R}$ for all $r \geq 2$.

We say that a sequence

$$F_1 \hookrightarrow F_2 \hookrightarrow F_3 \leftrightarrow \ldots$$

of embeddings of locally compact groups is stable (resp. weakly stable) if for every degree $q$ there exists $R_0 \in \mathbb{N}$ such that

$$H^q_{cb}(F_{R_0}) \leftrightarrow H^q_{cb}(F_{R_0+1}) \leftrightarrow H^q_{cb}(F_{R_0+2}) \leftrightarrow \ldots$$

consists of isomorphisms (resp. injections).

**Theorem 2.2** (De la Cruz Mengual, Hartnick). The families $O_{2r+1}(C)$, $Sp_{2r}(C)$, and $O_{2r}(C)$, together with block embeddings, are weakly stable with stability range $R_0$ of order $O(2^q)$.
A quantitative version of this theorem yields a weak stability range $R_0 = 12$ for $q = 3$. Kastenholz and Sroka [3] improved the stability range to be of linear order with respect to $q$ and obtain $R_0 = 6$ for $q = 3$. However, nothing is known about the bounded cohomology of $O_{13}(\mathbb{C})$, $Sp_{12}(\mathbb{C})$, or $O_{12}(\mathbb{C})$.

3 Bootstrapping

The crucial technique is bootstrapping an arbitrary stability range to the optimal one. From now on, we exclusively consider the symplectic family $Sp_{2r}(\mathbb{C})$.

**Key Lemma 3.1.** For every $r \geq 1$, the induced map

$$\iota_r^*: H^3_{cb}(Sp_{2r+2}(\mathbb{C})) \rightarrow H^3_{cb}(Sp_{2r}(\mathbb{C}))$$

is injective.

**Lemma 3.2.** For every $r \geq 1$, there exists a linear isomorphism

$$\ker(\iota_r^*) \cong H^3_{cb}(Sp_{2r+2}(\mathbb{C}) \smallfrown \mathbb{P}(V_{r+1})).$$

**Lemma 3.3.** For every $r \geq 1$, there exists an isomorphism

$$H^3_{cb}(Sp_{2r+2}(\mathbb{C}) \smallfrown \mathbb{P}(V_{r+1})) \cong H^3_{cb}(Sp_{2r}(\mathbb{C}) \smallfrown \mathbb{P}(V_2)).$$

**Proof of Key Lemma.** We have $\ker(\iota_r^*) \cong H^3_{cb}(Sp_{4}(\mathbb{C}) \smallfrown \mathbb{P}(V_2))$. There exists $s \geq 1$ such that $\iota_s^*: H^3_{cb}(Sp_{2s+2}(\mathbb{C})) \rightarrow H^3_{cb}(Sp_{2s}(\mathbb{C}))$ is injective, so $\ker(\iota_r^*) = 0$ for every $r \geq 1$.

**Corollary 3.4.** For every $r \geq 1$, the comparison map $H^3_{cb}(Sp_{2r}(\mathbb{C})) \rightarrow H^3_{cb}(Sp_{2r}(\mathbb{C}))$ is an isomorphism.

**Proof.** We use induction on $r$. We have $Sp_2(\mathbb{C}) = SL_2(\mathbb{C})$, and the conjecture is already known for $SL_2(\mathbb{C})$.

Now we assume that $c^3: H^3_{cb}(Sp_{2r}(\mathbb{C})) \rightarrow H^3_{cb}(Sp_{2r}(\mathbb{C}))$ is an isomorphism. Let us consider the commutative diagram

$$\begin{array}{ccc}
H^3_{cb}(Sp_{2r+2}(\mathbb{C})) & \xrightarrow{c^3} & H^3_{cb}(Sp_{2r+2}(\mathbb{C})) \\
\downarrow & & \downarrow \\
H^3_{cb}(Sp_{2r}(\mathbb{C})) & \xrightarrow{c^3} & H^3_{cb}(Sp_{2r}(\mathbb{C}))
\end{array}$$

It is known that the right arrow is an isomorphism. Hence it suffices to show that $\iota_r^*$ is a linear isomorphism.

We already know that $\iota_r^*$ is injective. First, we inductively prove that the restriction map
res_\ast : H^3_{cb}(\text{SL}_2_r(\mathbb{C})) \to H^3_{cb}(\text{Sp}_{2r}(\mathbb{C})) is a linear isomorphism. The case \( r = 1 \) is trivial. Let us now assume that \( \text{res}_r \) is a linear isomorphism. From the commutative diagram

\[
\begin{array}{ccc}
H^3_{cb}(\text{SL}_{2r+2}(\mathbb{C})) & \xrightarrow{\text{res}_{r+1}} & H^3_{cb}(\text{Sp}_{2r+2}(\mathbb{C})) \\
\downarrow & & \downarrow \\
H^3_{cb}(\text{SL}_{2r}(\mathbb{C})) & \xrightarrow{\text{res}_r} & H^3_{cb}(\text{Sp}_{2r}(\mathbb{C}))
\end{array}
\]

we see that \( \iota^r_\ast \) is a linear isomorphism. Hence \( \text{res}_{r+1} \) is also a linear isomorphism. Thus, \( \iota^r_\ast \) is a linear isomorphism for every \( r \geq 1 \). \( \square \)

4 The Spectral Sequence

Let \( V_r \) denote the 2\( r \)-dimensional complex vector space with basis \{\( e_r, \ldots, e_1, f_1, \ldots, f_r \)\}. We define a symplectic form \( \omega : V_r \times V_r \to \mathbb{C} \) by setting \( \omega(e_i, f_i) = 1 \) for all \( i = 1, \ldots, r \). Then \( \text{Sp}_{2r}(\mathbb{C}) \) is the automorphism group of \( (V_r, \omega) \), and acts transitively on \( \mathbb{P}(V_r) \).

Consider the action

\[
\text{Sp}_{2r}(\mathbb{C}) \acts L^\infty(\mathbb{P}(V_r)^p), (g, f)(v_1, \ldots, v_p) := f(g^{-1}v_1, \ldots, g^{-1}v_p)
\]

and the complex

\[
0 \to \mathbb{R} \xrightarrow{d^{r-1}_r} L^\infty(\mathbb{P}(V_r)) \xrightarrow{d^p_r} L^\infty(\mathbb{P}(V_r)^2) \xrightarrow{d^1_r} L^\infty(\mathbb{P}(V_r)^3) \to \ldots
\]

where

\[
d^{r-1}_r : L^\infty(\mathbb{P}(V_r)) \to L^\infty(\mathbb{P}(V_r)^p),
\]

\[
d^{p-1}_r(f)(v_0, \ldots, v_p) := \sum_{j=0}^{p} (-1)^j f(v_0, \ldots, \hat{v}_j, \ldots, v_p)
\]

for all \( p \in \mathbb{N} \), and \( d^{-1}_r \) is the inclusion of constants. We denote by \( H^\ast_{cb}(\text{Sp}_{2r}(\mathbb{C}) \acts \mathbb{P}(V_r)) \) the homology of the complex \( L^\infty(\mathbb{P}(V_r)^{k+1})_{\text{Sp}_{2r}(\mathbb{C})} \).

Let \( Q_r \) be the stabilizer of the point \([e_r] \in \mathbb{P}(V_r)\). The Levi decomposition yields \( Q_r \cong U_r \times R_r \), where \( R_r \cong \mathbb{C}^\times \times \text{Sp}_{2r-2}(\mathbb{C}) \). We write

\[
\mathbb{P}(V_r)^{k+1} := \{v = [v_0, \ldots, v_k] \in \mathbb{P}(V_r)^{k+1} | \text{v is in general position, } \omega(v_i, v_j) \neq 0\}
\]

and denote the point stabilizer of the group action \( \text{Sp}_{2r}(\mathbb{C}) \acts \mathbb{P}(V_r)^{3} \) by \( S_r \). We have a short exact sequence

\[
1 \to N \to S_r \to \text{Sp}_{2r-4}(\mathbb{C}) \to 1,
\]

where \( N \) is solvable.

A spectral sequence \( E_r^{p, q} \bullet \) is a sequence of differential bigraded vector spaces, i.e. for \( r \in \mathbb{N} \) and \( p, q \in \mathbb{N}_0 \) we have a vector space \( E_r^{p, q} \) together with differentials

\[
d^{p, q}_r : E^{p, q}_r \to E^{p+r, q-r+1}_r
\]

such that \( E^{p, q}_{r+1} = \ker(d^{p, q}_r)/\text{im}(d^{p-r, q+r-1}_r) \). For \( r > \max(p, q + 1) \) we have \( E^{p, q}_r = E^{p, q}_r \), so we can define \( E^\infty := E^{p, q}_r \) and say that \( E_r^{p, q} \bullet \) converges to \( E^\infty \bullet \).
Proof of Lemma 3.2. We define $E_1^{p,q} = H^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}); L^\infty(\mathbb{P}(V_{r+1})^p))$ and the induced maps
\[
d_1^{p,q} : C^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}); L^\infty(\mathbb{P}(V_{r+1})^p)) \to C^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}); L^\infty(\mathbb{P}(V_{r+1})^{p+1})), \quad c \mapsto d_1^{p-1} \circ c
\]
and $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$. In De la Cruz Mengual and Hartnick [2], Proposition 2.15 it is shown that $E_1^{p,\bullet}$ converges to zero. We have
\[
E_1^{0,q} \cong H^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C})),
\]
\[
E_1^{1,q} \cong H^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}); I_{Q_{r+1}}^{\text{Sp}_{2r+2}(\mathbb{C})}(\mathbb{R})) \cong H^q_{cb}(Q_{r+1}) \cong H^q_{cb}(\text{Sp}_{2r}(\mathbb{C})),
\]
\[
E_1^{2,q} \cong H^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}); I_{R_{r+1}}^{\text{Sp}_{2r+2}(\mathbb{C})}(\mathbb{R})) \cong H^q_{cb}(R_{r+1}) \cong H^q_{cb}(\text{Sp}_{2r}(\mathbb{C})),
\]
\[
E_1^{3,q} \cong H^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}); I_{S_{r+1}}^{\text{Sp}_{2r+2}(\mathbb{C})}(\mathbb{R})) \cong H^q_{cb}(S_{r+1}) \cong H^q_{cb}(\text{Sp}_{2r-2}(\mathbb{C})).
\]
Hence $E_1^{p,1} = 0 = E_1^{p,2}$, $p \leq 3$. In De la Cruz Mengual and Hartnick [2], Lemma 3.7 it is shown that $d_1^{0,3} : E_1^{0,3} \to E_1^{1,3}$ is conjugated to $\iota_1^*$ by the isomorphisms above. Hence $E_2^{3,0} \cong \ker(\iota_1^*)$. We have $E_1^{p,0} = L^\infty(\mathbb{P}(V_{r+1})^p)^{\text{Sp}_{2r+2}(\mathbb{C})}$. Furthermore, we have $E_2^{p,0} = H^q_{cb}(\text{Sp}_{2r+2}(\mathbb{C}) \cap \mathbb{P}(V_{r+1}))$ for all $p \geq 0$. Also $E_2^{1,2} = E_2^{2,1} = E_2^{2,2} = E_2^{3,1} = 0$, and all these terms remain unchanged until the fourth page $E_4^{p,\bullet}$. Since $E_4^{p,\bullet}$ converges to zero, we know that
\[
0 = E_5^{0,3} = \ker(d_4^{0,3})/\text{im}(d_4^{-4,6}) = \ker(d_4^{0,3}),
\]
\[
0 = E_5^{4,0} = \ker(d_4^{4,0} : E_4^{4,0} \to E_4^{8,-3})/\text{im}(d_4^{0,3}) = E_4^{4,0}/\text{im}(d_4^{0,3}).
\]
Hence $\text{im}(d_4^{0,3}) = E_4^{1,0}$, so $d_4^{0,3} : E_4^{0,3} \to E_4^{4,0}$ is an isomorphism between $\ker(\iota_1^*)$ and $H^3_{cb}(\text{Sp}_{2r+2}(\mathbb{C}) \cap \mathbb{P}(V_{r+1}))$. \hfill \Box

Proposition 4.1. (a) $\text{Sp}_{2r}(\mathbb{C})$ acts transitively on $\mathbb{P}(V_r)^{[3]}$, $r \geq 1$.
(b) There are cross-ratios $\text{cr}_1$ and $\text{cr}_2$ such that the $\text{Sp}_{2r}(\mathbb{C})$-invariant map
\[
\pi_3 : \mathbb{P}(V_r)^{[4]} \to \mathbb{C}^2, \quad \pi_3 := (\text{cr}_1, \text{cr}_2)
\]
induces an isomorphism $\text{Sp}_{2r}(\mathbb{C}) \backslash \mathbb{P}(V_r)^{[4]} \cong \mathbb{C}^2$, $r \geq 2$.
(c) There are cross-ratios $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ such that the $\text{Sp}_{2r}(\mathbb{C})$-invariant map
\[
\pi_4 : \mathbb{P}(V_r)^{[5]} \to \mathbb{C}^5, \quad \pi_4 := (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)
\]
induces an isomorphism $\text{Sp}_{2r}(\mathbb{C}) \backslash \mathbb{P}(V_r)^{[5]} \cong \mathbb{C}^5$, $r \geq 2$. 

5
Proof of Lemma 3.3. We know

\[ L^\infty(\mathbb{P}(V_{r+1})^3)^{\text{Sp}_{2r+2}(\mathbb{C})} \cong L^\infty(\text{Sp}_{2r+2}(\mathbb{C}) \setminus \mathbb{P}(V_{r+1})^3) \cong \mathbb{R}, \]
\[ L^\infty(\mathbb{P}(V_{r+1})^4)^{\text{Sp}_{2r+2}(\mathbb{C})} \cong L^\infty(\text{Sp}_{2r+2}(\mathbb{C}) \setminus \mathbb{P}(V_{r+1})^4) \cong L^\infty(\mathbb{C}^2), \]
\[ L^\infty(\mathbb{P}(V_{r+1})^5)^{\text{Sp}_{2r+2}(\mathbb{C})} \cong L^\infty(\text{Sp}_{2r+2}(\mathbb{C}) \setminus \mathbb{P}(V_{r+1})^5) \cong L^\infty(\mathbb{C}^5). \]

Hence

\[ d^{3,0}_1 = d^{2,0}_{r+1} : L^\infty(\mathbb{P}(V_{r+1})^3)^{\text{Sp}_{2r+2}(\mathbb{C})} \to L^\infty(\mathbb{P}(V_{r+1})^4)^{\text{Sp}_{2r+2}(\mathbb{C})}, \]
\[ d^{3,0}_1(f)(v_0, v_1, v_2, v_3) = f(v_1, v_2, v_3) - f(v_0, v_2, v_3) + f(v_0, v_1, v_3) - f(v_0, v_1, v_2) = 0, \]

so

\[ H^3_{\text{cb}}(\text{Sp}_{2r+2}(\mathbb{C}) \lhd \mathbb{P}(V_{r+1})) = E^4_{2,0} = \ker(d^{4,0}_1)/\text{im}(d^{3,0}_1) = \ker(d^{3}_{r+1}). \]

Let \( D_{r+1}^3 \) be the operator making the following diagram commute:

\[ \begin{array}{ccc}
L^\infty(\mathbb{C}^2) & \xrightarrow{D_{r+1}^3} & L^\infty(\mathbb{C}^5) \\
\pi^3_1 \downarrow & & \downarrow \pi^3_2 \\
L^\infty(\mathbb{P}(V_{r+1})^4)^{\text{Sp}_{2r+2}(\mathbb{C})} & \xrightarrow{d^{3}_{r+1}} & L^\infty(\mathbb{P}(V_{r+1})^5)^{\text{Sp}_{2r+2}(\mathbb{C})}
\end{array} \]

One guesses

\[ D_{r+1}^3(f)(a_1, a_2, b_1, b_2, c) = f\left(\frac{-a_1 c}{a_2 b_1}, \frac{c}{a_2}, \frac{-b_2 c}{a_2 b_1}\right) - f\left(\frac{c}{b_1}, \frac{-b_2 c}{a_2 b_1}\right) + f\left(c, \frac{-a_1 b_2 c}{a_2 b_1}\right) - f(b_1, b_2) + f(a_1, a_2) \]

and checks that this makes the diagram above commute. Now we see that \( \ker(d^{3}_{r+1}) \cong \ker(D_{r+1}^3) \) is independent of \( r \), which yields the claim. \( \Box \)
References

