1 Introduction

Let G be a locally compact (Hausdorff) group. We call

$$C^n_{cb}(G) := \{f \in C(G^{n+1}, \mathbb{R}) \mid f(gg_0, \dots, gg_n) = f(g_0, \dots, g_n),\ \|f\|_\infty < \infty\}$$

the space of *continuous bounded homogeneous n-cochains* and define the *coboundary map* $d^{n-1} : C^{n-1}_{cb}(G) \to C^n_{cb}(G)$ by

$$d^{n-1}(f)(g_0, \dots, g_n) := \sum_{j=0}^{n} (-1)^j f(g_0, \dots, \widehat{g_j}, \dots, g_n).$$

This yields a cochain complex (i.e. $d^n \circ d^{n-1} = 0$), so we can consider its *n-th cohomology*

$$H^n_{cb}(G) := \ker(d^n)/\mathrm{im}(d^{n-1}).$$

We call $H^n_{cb}(G)$ the *n-th continuous bounded cohomology space* of G with $\mathbb{R}$-coefficients. We are interested in the continuous bounded cohomology of connected groups.

Theorem 1.1 (Monod, 2001)**.** Let G be a locally compact second countable group and let N be a closed normal amenable subgroup. Then $H^*_{cb}(G) \cong H^*_{cb}(G/N)$.

Theorem 1.2 (Gleason-Yamabe)**.** Let G be a connected locally compact group. For every open neighborhood U of the identity there exists a compact normal subgroup K of G in U such that G/K is isomorphic to a connected Lie group.

Corollary 1.3. Let G be a connected locally compact second countable group. Then there exists a connected semisimple center-free Lie group without compact factors G' such that $H^*_{cb}(G) \cong H^*_{cb}(G')$.

Proof. There exists a compact normal subgroup K of G such that G/K is isomorphic to a connected Lie group. We can write

$$\widetilde{G/K} \cong R \rtimes (S_1 \times S_2),$$

where R denotes the solvable radical of $\widetilde{G/K}$, S_1 is a connected compact semisimple Lie group and S_2 is a connected semisimple Lie group without compact factors. Since solvable groups and compact groups are amenable, this yields the isomorphism

$$H^*_{cb}(G) \cong H^*_{cb}(S_2/Z(S_2)).$$

Here, $G' = S_2/Z(S_2)$ is a connected semisimple center-free Lie group without compact factors. □

Conjecture 1.4 (Monod, 2006)**.** Let G be a connected semisimple finite-center Lie group without compact factors. Then the *comparison map*

$$H^*_{cb}(G) \to H^*_c(G),\ [c]_b \mapsto [c]$$

is an isomorphism.

Here, $H_c^*(G)$ denotes the *continuous cohomology* of G. It is known explicitly in all cases. Note that both continuous bounded cohomology and continuous cohomology are invariant under finite extensions. The comparison map is not an isomorphism for $\widetilde{\mathrm{SL}_2}(\mathbb{R})$, which has infinite center.
A Künneth formula is conjectured to hold for continuous bounded cohomology. Thus, it would suffice to check the isomorphism conjecture for connected simple center-free Lie groups without compact factors. These groups are classified.
Degree 0/1: Not difficult.
Degree 2: Proven by Burger and Monod.
Degree 3: Proven for $\mathrm{SL}_n(\mathbb{R})$ by Burger and Monod. Proven for $\mathrm{SL}_n(\mathbb{C})$ by Goncharov, Bloch, and Monod. Recently proven for the groups $\mathrm{SO}_{2r+1}(\mathbb{C})$, $\mathrm{Sp}_{2r}(\mathbb{C})$, and $\mathrm{SO}_{2r}(\mathbb{C})$ by De la Cruz Mengual and Hartnick.
Degree 4: Proven for $\mathrm{SL}_2(\mathbb{R})$ by Hartnick and Ott.
We will focus on the recent results by De la Cruz Mengual and Hartnick, see [1].

2 Bounded-Cohomological Stabilization

Fact 2.1. Let G be a connected simple finite-center Lie group. Then

$$H_c^3(G) \neq 0 \iff G \text{ is a complex Lie group.}$$

In that case, $H_c^3(G) \cong \mathbb{R}$ is generated by the *Borel class.*

Consider the block embeddings

$$\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{SL}_3(\mathbb{C}) \hookrightarrow \mathrm{SL}_4(\mathbb{C}) \hookrightarrow \dots$$

Monod proved that the induced sequence

$$H_{cb}^3(\mathrm{SL}_2(\mathbb{C})) \leftarrow H_{cb}^3(\mathrm{SL}_3(\mathbb{C})) \leftarrow H_{cb}^3(\mathrm{SL}_4(\mathbb{C})) \leftarrow \dots$$

consists of injections preserving the *bounded Borel class.* Combined with the fact that the comparison map $H_{cb}^3(\mathrm{SL}_2(\mathbb{C})) \to H_c^3(\mathrm{SL}_2(\mathbb{C}))$ is an isomorphism mapping the bounded Borel class to the Borel class, this implies $H_{cb}^3(\mathrm{SL}_r(\mathbb{C})) \cong \mathbb{R}$ for all $r \geq 2$.

We say that a sequence

$$F_1 \hookrightarrow F_2 \hookrightarrow F_3 \hookrightarrow \dots$$

of embeddings of locally compact groups is *stable* (resp. *weakly stable*) if for every degree q there exists $R_0 \in \mathbb{N}$ such that

$$H_{cb}^q(F_{R_0}) \leftarrow H_{cb}^q(F_{R_0+1}) \leftarrow H_{cb}^q(F_{R_0+2}) \leftarrow \dots$$

consists of isomorphisms (resp. injections).

Theorem 2.2 (De la Cruz Mengual, Hartnick)**.** The families $\mathrm{O}_{2r+1}(\mathbb{C})$, $\mathrm{Sp}_{2r}(\mathbb{C})$, and $\mathrm{O}_{2r}(\mathbb{C})$, together with block embeddings, are weakly stable with stability range R_0 of order $O(2^q)$.

A quantitative version of this theorem yields a weak stability range $R_0 = 12$ for $q = 3$. Kastenholz and Sroka [3] improved the stability range to be of linear order with respect to q and obtain $R_0 = 6$ for $q = 3$. However, nothing is known about the bounded cohomology of $\mathrm{O}_{13}(\mathbb{C})$, $\mathrm{Sp}_{12}(\mathbb{C})$, or $\mathrm{O}_{12}(\mathbb{C})$.

3 Bootstrapping

The crucial technique is bootstrapping an arbitrary stability range to the *optimal* one. From now on, we exclusively consider the symplectic family $\mathrm{Sp}_{2r}(\mathbb{C})$.

Key Lemma 3.1. For every $r \geq 1$, the induced map

$$\iota_r^* : H^3_{cb}(\mathrm{Sp}_{2r+2}(\mathbb{C})) \to H^3_{cb}(\mathrm{Sp}_{2r}(\mathbb{C}))$$

is injective.

Lemma 3.2. For every $r \geq 1$, there exists a linear isomorphism

$$\ker(\iota_r^*) \cong H^3_{cb}(\mathrm{Sp}_{2r+2}(\mathbb{C}) \curvearrowright \mathbb{P}(V_{r+1})).$$

Lemma 3.3. For every $r \geq 1$, there exists an isomorphism

$$H^3_{cb}(\mathrm{Sp}_{2r+2}(\mathbb{C}) \curvearrowright \mathbb{P}(V_{r+1})) \cong H^3_{cb}(\mathrm{Sp}_4(\mathbb{C}) \curvearrowright \mathbb{P}(V_2)).$$

Proof of Key Lemma. We have $\ker(\iota_r^*) \cong H^3_{cb}(\mathrm{Sp}_4(\mathbb{C}) \curvearrowright \mathbb{P}(V_2))$. There exists $s \geq 1$ such that $\iota_s^* : H^3_{cb}(\mathrm{Sp}_{2s+2}(\mathbb{C})) \to H^3_{cb}(\mathrm{Sp}_{2s}(\mathbb{C}))$ is injective, so $\ker(\iota_r^*) = 0$ for every $r \geq 1$. □

Corollary 3.4. For every $r \geq 1$, the comparison map $H^3_{cb}(\mathrm{Sp}_{2r}(\mathbb{C})) \to H^3_c(\mathrm{Sp}_{2r}(\mathbb{C}))$ is an isomorphism.

Proof. We use induction on r. We have $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$, and the conjecture is already known for $\mathrm{SL}_2(\mathbb{C})$.
Now we assume that $c^3 : H^3_{cb}(\mathrm{Sp}_{2r}(\mathbb{C})) \to H^3_c(\mathrm{Sp}_{2r}(\mathbb{C}))$ is an isomorphism. Let us consider the commutative diagram

$$\begin{array}{ccc} H^3_{cb}(\mathrm{Sp}_{2r+2}(\mathbb{C})) & \xrightarrow{c^3} & H^3_c(\mathrm{Sp}_{2r+2}(\mathbb{C})) \\ \downarrow{\scriptstyle \iota_r^*} & & \downarrow \\ H^3_{cb}(\mathrm{Sp}_{2r}(\mathbb{C})) & \xrightarrow[c^3]{} & H^3_c(\mathrm{Sp}_{2r}(\mathbb{C})) \end{array}$$

It is known that the right arrow is an isomorphism. Hence it suffices to show that ι_r^* is a linear isomorphism.
We already know that ι_r^* is injective. First, we inductively prove that the restriction map

$\mathrm{res}_r : H^3_{cb}(\mathrm{SL}_{2r}(\mathbb{C})) \to H^3_{cb}(\mathrm{Sp}_{2r}(\mathbb{C}))$ is a linear isomorphism. The case $r = 1$ is trivial. Let us now assume that res_r is a linear isomorphism. From the commutative diagram

$$\begin{array}{ccc} H^3_{cb}(\mathrm{SL}_{2r+2}(\mathbb{C})) & \xrightarrow{\mathrm{res}_{r+1}} & H^3_{cb}(\mathrm{Sp}_{2r+2}(\mathbb{C})) \\ \cong\downarrow & & \downarrow \iota_r^* \\ H^3_{cb}(\mathrm{SL}_{2r}(\mathbb{C})) & \xrightarrow[\mathrm{res}_r]{} & H^3_{cb}(\mathrm{Sp}_{2r}(\mathbb{C})) \end{array}$$

we see that ι_r^* is a linear isomorphism. Hence res_{r+1} is also a linear isomorphism. Thus, ι_r^* is a linear isomorphism for every $r \geq 1$. □

4 The Spectral Sequence

Let V_r denote the $2r$-dimensional complex vector space with basis $\{e_r, \ldots, e_1, f_1, \ldots, f_r\}$. We define a symplectic form $\omega : V_r \times V_r \to \mathbb{C}$ by setting $\omega(e_i, f_i) = 1$ for all $i = 1, \ldots, r$. Then $\mathrm{Sp}_{2r}(\mathbb{C})$ is the automorphism group of (V_r, ω), and acts transitively on $\mathbb{P}(V_r)$. Consider the action

$$\mathrm{Sp}_{2r}(\mathbb{C}) \curvearrowright L^\infty(\mathbb{P}(V_r)^p),\ (g.f)(v_1, \ldots, v_p) := f(g^{-1}.v_1, \ldots, g^{-1}.v_p)$$

and the complex

$$0 \to \mathbb{R} \xrightarrow{d_r^{-1}} L^\infty(\mathbb{P}(V_r)) \xrightarrow{d_r^0} L^\infty(\mathbb{P}(V_r)^2) \xrightarrow{d_r^1} L^\infty(\mathbb{P}(V_r)^3) \to \ldots$$

where

$$d_r^{p-1} : L^\infty(\mathbb{P}(V_r)^p) \to L^\infty(\mathbb{P}(V_r)^{p+1}),$$
$$d_r^{p-1}(f)(v_0, \ldots, v_p) := \sum_{j=0}^{p} (-1)^j f(v_0, \ldots, \widehat{v_j}, \ldots, v_p)$$

for all $p \in \mathbb{N}$, and d_r^{-1} is the inclusion of constants. We denote by $H^*_{cb}(\mathrm{Sp}_{2r}(\mathbb{C}) \curvearrowright \mathbb{P}(V_r))$ the homology of the complex $L^\infty(\mathbb{P}(V_r)^{*+1})^{\mathrm{Sp}_{2r}(\mathbb{C})}$.

Let Q_r be the stabilizer of the point $[e_r] \in \mathbb{P}(V_r)$. The Levi decomposition yields $Q_r \cong U_r \rtimes R_r$, where $R_r \cong \mathbb{C}^\times \times \mathrm{Sp}_{2r-2}(\mathbb{C})$. We write

$$\mathbb{P}(V_r)^{\{k+1\}} := \{v = [v_0, \ldots, v_k] \in \mathbb{P}(V_r)^{k+1} \mid v \text{ is in general position, } \omega(v_i, v_j) \neq 0\}$$

and denote the point stabilizer of the group action $\mathrm{Sp}_{2r}(\mathbb{C}) \curvearrowright \mathbb{P}(V_r)^{\{3\}}$ by S_r. We have a short exact sequence

$$1 \to N \to S_r \to \mathrm{Sp}_{2r-4}(\mathbb{C}) \to 1,$$

where N is solvable.

A *spectral sequence* $E_\bullet^{\bullet,\bullet}$ is a sequence of differential bigraded vector spaces, i.e. for $r \in \mathbb{N}$ and $p, q \in \mathbb{N}_0$ we have a vector space $E_r^{p,q}$ together with differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $E_{r+1}^{p,q} = \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,q+r-1})$. For $r > \max(p, q+1)$ we have $E_{r+1}^{p,q} = E_r^{p,q}$, so we can define $E_\infty^{p,q} := E_r^{p,q}$ and say that $E_\bullet^{\bullet,\bullet}$ *converges* to $E_\infty^{\bullet,\bullet}$.

Proof of Lemma 3.2. We define $E_1^{p,q} = H_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C}); L^\infty(\mathbb{P}(V_{r+1})^p))$ and the induced maps

$$d_1^{p,q} : C_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C}); L^\infty(\mathbb{P}(V_{r+1})^p)) \to C_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C}); L^\infty(\mathbb{P}(V_{r+1})^{p+1})),\ c \mapsto d^{p-1} \circ c$$

and $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$. In De la Cruz Mengual and Hartnick [2], Proposition 2.15 it is shown that $E_\bullet^{\bullet,\bullet}$ converges to zero. We have

$$E_1^{0,q} \cong H_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C})),$$

$$\begin{aligned} E_1^{1,q} &\cong H_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C}); I_{Q_{r+1}}^{\mathrm{Sp}_{2r+2}(\mathbb{C})}(\mathbb{R})) \\ &\overset{ESL}{\cong} H_{cb}^q(Q_{r+1}) \cong H_{cb}^q(\mathrm{Sp}_{2r}(\mathbb{C})), \end{aligned}$$

$$\begin{aligned} E_1^{2,q} &\cong H_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C}); I_{R_{r+1}}^{\mathrm{Sp}_{2r+2}(\mathbb{C})}(\mathbb{R})) \\ &\overset{ESL}{\cong} H_{cb}^q(R_{r+1}) \cong H_{cb}^q(\mathrm{Sp}_{2r}(\mathbb{C})), \end{aligned}$$

$$\begin{aligned} E_1^{3,q} &\cong H_{cb}^q(\mathrm{Sp}_{2r+2}(\mathbb{C}); I_{S_{r+1}}^{\mathrm{Sp}_{2r+2}(\mathbb{C})}(\mathbb{R})) \\ &\overset{ESL}{\cong} H_{cb}^q(S_{r+1}) \cong H_{cb}^q(\mathrm{Sp}_{2r-2}(\mathbb{C})). \end{aligned}$$

Hence $E_1^{p,1} = 0 = E_1^{p,2}$, $p \leq 3$. In De la Cruz Mengual and Hartnick [2], Lemma 3.7 it is shown that $d_1^{0,3} : E_1^{0,3} \to E_1^{1,3}$ is conjugated to ι_r^* by the isomorphisms above. Hence $E_2^{0,3} \cong \ker(\iota_r^*)$. We have $E_1^{p,0} = L^\infty(\mathbb{P}(V_{r+1})^p)^{\mathrm{Sp}_{2r+2}(\mathbb{C})}$. Furthermore, we have $E_2^{p,0} = H_{cb}^{p-1}(\mathrm{Sp}_{2r+2}(\mathbb{C}) \curvearrowright \mathbb{P}(V_{r+1}))$ for all $p \geq 0$. Also $E_2^{1,2} = E_2^{2,1} = E_2^{2,2} = E_2^{3,1} = 0$, and all these terms remain unchanged until the fourth page $E_4^{\bullet,\bullet}$. Since $E_\bullet^{\bullet,\bullet}$ converges to zero, we know that

$$\begin{aligned} 0 &= E_5^{0,3} = \ker(d_4^{0,3})/\mathrm{im}(d_4^{-4,6}) = \ker(d_4^{0,3}), \\ 0 &= E_5^{4,0} = \ker(d_4^{4,0} : E_4^{4,0} \to E_4^{8,-3})/\mathrm{im}(d_4^{0,3}) = E_4^{4,0}/\mathrm{im}(d_4^{0,3}). \end{aligned}$$

Hence $\mathrm{im}(d_4^{0,3}) = E_4^{4,0}$, so $d_4^{0,3} : E_4^{0,3} \to E_4^{4,0}$ is an isomorphism between $\ker(\iota_r^*)$ and $H_{cb}^3(\mathrm{Sp}_{2r+2}(\mathbb{C}) \curvearrowright \mathbb{P}(V_{r+1}))$. □

Proposition 4.1. (a) $\mathrm{Sp}_{2r}(\mathbb{C})$ acts transitively on $\mathbb{P}(V_r)^{\{3\}}$, $r \geq 1$.
(b) There are cross-ratios cr_1 and cr_2 such that the $\mathrm{Sp}_{2r}(\mathbb{C})$-invariant map

$$\pi_3 : \mathbb{P}(V_r)^{\{4\}} \to \mathbb{C}^2,\ \pi_3 := (\mathrm{cr}_1, \mathrm{cr}_2)$$

induces an isomorphism $\mathrm{Sp}_{2r}(\mathbb{C})\backslash\mathbb{P}(V_r)^{\{4\}} \cong \mathbb{C}^2$, $r \geq 2$.
(c) There are cross-ratios $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ such that the $\mathrm{Sp}_{2r}(\mathbb{C})$-invariant map

$$\pi_4 : \mathbb{P}(V_r)^{\{5\}} \to \mathbb{C}^5,\ \pi_4 := (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)$$

induces an isomorphism $\mathrm{Sp}_{2r}(\mathbb{C})\backslash\mathbb{P}(V_r)^{\{5\}} \cong \mathbb{C}^5$, $r \geq 2$.

Proof of Lemma 3.3. We know

$$\begin{aligned} L^\infty(\mathbb{P}(V_{r+1})^3)^{\mathrm{Sp}_{2r+2}(\mathbb{C})} &\cong L^\infty(\mathrm{Sp}_{2r+2}(\mathbb{C})\backslash\mathbb{P}(V_{r+1})^3) \cong \mathbb{R}, \\ L^\infty(\mathbb{P}(V_{r+1})^4)^{\mathrm{Sp}_{2r+2}(\mathbb{C})} &\cong L^\infty(\mathrm{Sp}_{2r+2}(\mathbb{C})\backslash\mathbb{P}(V_{r+1})^4) \cong L^\infty(\mathbb{C}^2), \\ L^\infty(\mathbb{P}(V_{r+1})^5)^{\mathrm{Sp}_{2r+2}(\mathbb{C})} &\cong L^\infty(\mathrm{Sp}_{2r+2}(\mathbb{C})\backslash\mathbb{P}(V_{r+1})^5) \cong L^\infty(\mathbb{C}^5). \end{aligned}$$

Hence

$$\begin{aligned} &d_1^{3,0} = d_{r+1}^2 : L^\infty(\mathbb{P}(V_{r+1})^3)^{\mathrm{Sp}_{2r+2}(\mathbb{C})} \to L^\infty(\mathbb{P}(V_{r+1})^4)^{\mathrm{Sp}_{2r+2}(\mathbb{C})}, \\ &d_1^{3,0}(f)(v_0,v_1,v_2,v_3) = f(v_1,v_2,v_3) - f(v_0,v_2,v_3) + f(v_0,v_1,v_3) - f(v_0,v_1,v_2) = 0, \end{aligned}$$

so

$$H^3_{cb}(\mathrm{Sp}_{2r+2}(\mathbb{C}) \curvearrowright \mathbb{P}(V_{r+1})) = E_2^{4,0} = \ker(d_1^{4,0})/\mathrm{im}(d_1^{3,0}) = \ker(d_{r+1}^3).$$

Let D^3_{r+1} be the operator making the following diagram commute:

$$\begin{array}{ccc} L^\infty(\mathbb{C}^2) & \xrightarrow{D^3_{r+1}} & L^\infty(\mathbb{C}^5) \\ \pi_3^* \big\downarrow & & \big\downarrow \pi_4^* \\ L^\infty(\mathbb{P}(V_{r+1})^4)^{\mathrm{Sp}_{2r+2}(\mathbb{C})} & \xrightarrow[d^3_{r+1}]{} & L^\infty(\mathbb{P}(V_{r+1})^5)^{\mathrm{Sp}_{2r+2}(\mathbb{C})} \end{array}$$

One guesses

$$\begin{aligned} &D^3_{r+1}(f)(a_1,a_2,b_1,b_2,c) \\ =&f(\frac{-a_1c}{a_2b_1},\frac{c}{a_2}) - f(\frac{c}{b_1},\frac{-b_2c}{a_2b_1}) + f(c,\frac{-a_1b_2c}{a_2b_1}) - f(b_1,b_2) + f(a_1,a_2) \end{aligned}$$

and checks that this makes the diagram above commute. Now we see that $\ker(d^3_{r+1}) \cong \ker(D^3_{r+1})$ is independent of r, which yields the claim. $\square$

References

[1] De la Cruz Mengual, Carlos. *The Degree-Three Bounded Cohomology of Complex Lie Groups of Classical Type.* arXiv:2304.00607.

[2] De la Cruz Mengual, Carlos, and Tobias Hartnick. *A Quillen stability criterion for bounded cohomology.* arXiv:2307.12808.

[3] Kastenholz, Thorben, and Robin J. Sroka. *Simplicial bounded cohomology and stability.* arXiv:2309.05024.