Let (X, d) be a metric space. A *geodesic ray* in X is a map $c : [0, \infty) \to X$ such that $d(c(s), c(t)) = |s - t|$ for all $s, t \geq 0$. We call two geodesic rays c_1 and c_2 in X *asymptotic* if there exists $K \geq 0$ such that $d(c_1(t), c_2(t)) \leq K$ for all $t \geq 0$. This is an equivalence relation. The equivalence class of a geodesic ray c is denoted by $c(\infty)$. The quotient set

$$\partial X = \{c(\infty) \mid c \text{ geodesic ray in } X\}$$

is called the (*visual*) *boundary* or the set of *points at infinity* of X. We write $\overline{X} = X \sqcup \partial X$. Every geodesic ray can be extended to a map $c : [0, \infty] \to \overline{X}$. If $\gamma \in \mathrm{Isom}(X)$, then $c_1(\infty) = c_2(\infty)$ if and only if $(\gamma \circ c_1)(\infty) = (\gamma \circ c_2)(\infty)$. Hence we can define $\gamma(c(\infty)) := (\gamma \circ c)(\infty)$, and we obtain a bijective map $\gamma : \overline{X} \to \overline{X}$.

Let X be a complete CAT(0) space, i.e. a Hadamard space. Then X is uniquely geodesic. The geodesic from $x \in X$ to $y \in X$ is denoted by $c_{x,y}$ and its image is denoted by $[x, y]$. Let $r \geq 0$. If $y \notin B(x, r)$, then

$$[x, y] \cap \partial_X B(x, r) = \{c_{x,y}(r)\},$$

so we can define

$$p_{x,r} : X \to \overline{B}(x, r),\ y \mapsto \begin{cases} y, & y \in B(x, r), \\ c_{x,y}(r), & y \notin B(x, r). \end{cases}$$

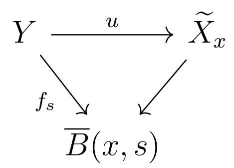
Note that $p_{x,r}|_{\overline{B}(x,r)} = \mathrm{id}_{\overline{B}(x,r)}$ and $p_{x,r}|_{\overline{B}(x,s)} \circ p_{x,s}|_{\overline{B}(x,t)} = p_{x,r}|_{\overline{B}(x,t)}$ for all $0 \leq r \leq s \leq t$. Hence $\{p_{x,r}|_{\overline{B}(x,s)} \mid 0 \leq r \leq s\}$ is an inverse system and we can consider the inverse limit $\varprojlim_{r \geq 0} \overline{B}(x, r)$.

Proposition 1. We have

$$\varprojlim_{r \geq 0} \overline{B}(x, r) = \widetilde{X}_x := \{c : [0, \infty) \to X \mid p_{x,r}(c(s)) = c(r),\ 0 \leq r \leq s\}$$

with the topology of uniform convergence on compact subsets, the maps $\widetilde{X}_x \to \overline{B}(x, s)$ being given by $c \mapsto c(s)$.

Proof. It is clear that $(\widetilde{X}_x \to \overline{B}(x, s))_{s \geq 0}$ is a cone. Let $(f_s : Y \to \overline{B}(x, s))_{s \geq 0}$ be another cone. Defining $u : Y \to \widetilde{X}_x$ by $u(y)(s) := f_s(y)$ makes the following diagram commute.



Of course, u is unique with this property. $\square$

If $c \in \widetilde{X}_x$, then either there exists $r_0 \geq 0$ such that $c(r) = c(r_0)$ for all $r \geq r_0$ and $c|_{[0,r_0]}$ is a geodesic from x to $c(r_0)$, or c is a geodesic ray issuing from x.

Proposition 2. There exists a natural bijection $\varphi_x : \overline{X} \to \widetilde{X}_x$.

Proof. We define $\varphi_x(y)$ to be the unique geodesic from x to y on $[0, d(x,y)]$ and to be equal to y on $(d(x,y), \infty)$ for all $y \in X$. Now let $c(\infty) \in \partial X$. We claim that there exists a unique geodesic ray c_x issuing from x such that $c_x(\infty) = c(\infty)$. Then we will be able to define $\varphi_x(c(\infty)) := c_x$.
Uniqueness: Let c_x, c'_x be two geodesic rays issuing from x that satisfy $c_x(\infty) = c(\infty) = c'_x(\infty)$. The function $[0,\infty) \to \mathbb{R}$, $t \mapsto d(c_x(t), c'_x(t))$ is non-negative, bounded, convex, and vanishes at zero. Thus, it is the zero function.
Existence: Let σ_t denote the geodesic from x to $c(t)$. For every $s \geq 0$ there exists some $n_s \in \mathbb{N}$ such that $\sigma_n(s)$ is defined for all $n \geq n_s$, so we can consider the sequence $(\sigma_n(s))_{n \geq n_s}$. One can show that this sequence is Cauchy, and we denote its limit by $c_x(s)$. It is then easy to see that c_x is the desired geodesic ray.
Naturality: In order to state the naturality claim, we first have to show that every $\gamma \in \mathrm{Isom}(X)$ extends to a homeomorphism $\widetilde{X}_x \to \widetilde{X}_{\gamma(x)}$. Let $y \in X$. Since isometries preserve geodesics, we have $\gamma(c_{x,y}(s)) = c_{\gamma(x),\gamma(y)}(s)$ for all $s \in [0, d(x,y)]$. Thus, the following diagram commutes for all $0 \leq s \leq t$.

$$\begin{array}{ccc} & \widetilde{X}_x & \\ \swarrow & & \searrow \\ \overline{B}(x,t) & \xrightarrow{p_{x,s}} & \overline{B}(x,s) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \overline{B}(\gamma(x),t) & \xrightarrow[p_{\gamma(x),s}]{} & \overline{B}(\gamma(x),s) \end{array}$$

By the universal property of the inverse limit, we obtain an induced map $\widetilde{X}_x \to \widetilde{X}_{\gamma(x)}$. It is easy to show that this map is a homeomorphism and is given by $c \mapsto \gamma \circ c$. Now naturality just means that the following diagram commutes.

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\gamma_*} & \overline{X} \\ {\scriptstyle\varphi_x}\downarrow & & \downarrow{\scriptstyle\varphi_{\gamma(x)}} \\ \widetilde{X}_x & \xrightarrow[\gamma_*]{} & \widetilde{X}_{\gamma(x)} \end{array}$$

This just means that $\gamma \circ c_{x,y} = c_{\gamma(x),\gamma(y)}$ and $\gamma \circ c_x = (\gamma \circ c)_{\gamma(x)}$. □

Transport of structure turns $\overline{X}$ into a topological space.

Corollary 3. The topology on $\overline{X}$ does not depend on the base point x.

Example 4. Geodesic rays in $\mathbb{R}^n$ are straight lines, and they are asymptotic if and only if they are parallel. Hence $\partial\mathbb{R}^n \cong \mathbb{S}^{n-1}$ and $\overline{\mathbb{R}^n} \cong \overline{B}(0,1)$ as topological spaces.

Example 5. We consider the Poincaré ball model for $\mathbb{H}^n$. The image of a geodesic ray $c : [0,\infty) \to \mathbb{H}^n$ is an arc of a circle or line that is orthogonal to the unit sphere. The closure of this arc has one endpoint on the sphere and $c(t)$ converges to this point as $t \to \infty$. Two rays are asymptotic if and only if the closures of their images intersect the sphere at the same point. Thus the boundary $\partial\mathbb{H}^n$ is naturally identified with $\mathbb{S}^{n-1}$ and $\overline{\mathbb{H}^n}$ is homeomorphic to $\overline{B}(0,1)$ in $\mathbb{R}^n$.

Example 6. More generally, if X is a simply connected complete Riemannian manifold of dimension n with non-positive cross-sectional curvature, then the map $T_x \to \partial X$ associating to each unit vector u tangent to X at x the class of the geodesic ray c which issues from x with velocity vector u.

An $\mathbb{R}$-tree is a uniquely geodesic metric space X such that $[y,x] \cup [x,z] = [y,z]$ if $[y,x] \cap [x,z] = \{x\}$. $\mathbb{R}$-trees are CAT(0) spaces.

Example 7. Let X be an $\mathbb{R}$-tree. Then ∂X is totally disconnected. If X is locally compact, then ∂X is compact. If X is an infinite simplicial $\mathbb{R}$-tree in which every vertex has valence at least three, then ∂X is homeomorphic to the Cantor set.