Boundary and Eigenvalue Problems

1. exercises sheet

Exercise 1

For which $1 \leq p \leq \infty$ does the following function belong to $W^{1,p}(U)$:

1. $U = [0, 2]$ and
\[ u(x) = \begin{cases} x & \text{for } x < 1 \\ 1 & \text{for } x \geq 1 \end{cases} \]

2. $U = [0, 2]$ and
\[ v(x) = \begin{cases} x & \text{for } x < 1 \\ 2 & \text{for } x \geq 1 \end{cases} \]

3. $U = B_1(0) \subset \mathbb{R}^n$ and
\[ w(x) = \sum_{i=1}^{\infty} 2^{-i} |x - x_i|^{-\alpha} \text{ for } \alpha > 0 \text{ and } \{x_i\}_{i \in \mathbb{N}} \subset U \text{ countable dense.} \]

Exercise 2

1. Let $\{v_m\}_{m \in \mathbb{N}} \subset \mathbb{R}^n$ be any sequence with $\lim_{m \to \infty} v_m = 0$. Show that for any $f \in W^{k,p}(\mathbb{R}^n)$ we have
\[ f_m(x) := f(x + v_m) \to f \text{ in } W^{k,p}(\mathbb{R}^n). \]

2. Let $g \in C^1_c(B_1)$ with $\|g\|_{L^p(\mathbb{R}^n)} > 0$ and $\{w_m\}_{m \in \mathbb{N}} \subset \mathbb{R}^n$ with $\lim_{m \to \infty} |w_m| = \infty$ be given. Show that no subsequence of $g_m(x) := g(x + w_m)$ can converge strongly in $L^p(\mathbb{R}^n)$.

Exercise 3

1. Suppose $U \subset \mathbb{R}^n$ connected and $f \in W^{1,p}(U)$ satisfies
\[ Df = 0 \text{ a.e. in } U. \]

Show that $f$ is constant a.e. in $U$.

2. Suppose $V \subset \mathbb{R}^n$ open and $u \in W^{1,p}(V)$, $v \in W^{1,q}(V)$. Show that $uv \in W^{1,r}(V)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and
\[ D(uv) = u Dv + Du v. \]

Exercise 4

Let $\eta_\epsilon$ be the usual mollifier. Given $u \in W^{1,p}(\mathbb{R}^n)$ and $u_\epsilon := \eta_\epsilon \ast u$. Show that
\[ \|u_\epsilon - u\|_{L^p(\mathbb{R}^n)} \leq \epsilon \|Du\|_{L^p(\mathbb{R}^n)}. \]

Furthermore conclude that if $\{u_m\}_{m \in \mathbb{N}} \subset W^{1,p}(\mathbb{R}^n)$ is a uniformly bounded sequence, then $\eta_\epsilon \ast u_m \to u_m$ uniformly as $\epsilon \to 0$. 

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