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Boundary and Eigenvalue Problems

1. exercises sheet

Exercise 1

For which $1 \leq p \leq \infty$ does the following function belong to $W^{1,p}(U)$:

1. $U =]0,2[$ and
$$u(x) = \begin{cases} x & \text{for } x < 1 \\ 1 & \text{for } x \geq 1 \end{cases};$$
2. $U =]0,2[$ and
$$v(x) = \begin{cases} x & \text{for } x < 1 \\ 2 & \text{for } x \geq 1 \end{cases};$$
3. $U = B_1(0) \subset \mathbb{R}^n$ and
$$w(x) = \sum_{i=1}^{\infty} 2^{-i}|x - x_i|^{-\alpha} \text{ for } \alpha > 0 \text{ and } \{x_i\}_{i\in\mathbb{N}} \subset U \text{ countable dense.}$$

Exercise 2

1. Let $\{v_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^n$ be any sequence with $\lim_{m\to\infty} v_m = 0$. Show that for any $f \in W^{k,p}(\mathbb{R}^n)$ we have
$$f_m(x) := f(x + v_m) \to f \text{ in } W^{k,p}(\mathbb{R}^n).$$
2. Let $g \in C_c^1(B_1)$ with $\|g\|_{L^p(\mathbb{R}^n)} > 0$ and $\{w_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^n$ with $\lim_{m\to\infty}|w_m| = \infty$ be given. Show that no subsequence of $g_m(x) := g(x + w_m)$ can converge strongly in $L^p(\mathbb{R}^n)$.

Exercise 3

1. Suppose $U \subset \mathbb{R}^n$ connected and $f \in W^{1,p}(U)$ satisfies
$$Df = 0 \text{ a.e. in } U.$$
Show that f is constant a.e. in U.
2. Suppose $V \subset \mathbb{R}^n$ open and $u \in W^{1,p}(V)$, $v \in W^{1,q}(V)$. Show that $u\,v \in W^{1,r}(V)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and
$$D(uv) = u\,Dv + Du\,v.$$

Exercise 4

Let η_ϵ be the usual mollifier. Given $u \in W^{1,p}(\mathbb{R}^n)$ and $u_\epsilon := \eta_\epsilon * u$. Show that
$$\|u_\epsilon - u\|_{L^p(\mathbb{R}^n)} \leq \epsilon \|Du\|_{L^p(\mathbb{R}^n)}.$$
Furthermore conclude that if $\{u_m\}_{m\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^n)$ is a uniformly bounded sequence, then $\eta_\epsilon * u_m \to u_m$ uniformly as $\epsilon \to 0$.