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Dr. Ioannis Anapolitanos
Dr. Jonas Hirsch

Boundary and Eigenvalue Problems

10. exercise sheet

On this sheet we assume that all coefficient are smooth, uniformly bounded functions on a domain $U \subset \mathbb{R}^n$ with smooth boundary. Furthermore we assume that the coefficients $a^{ij} = a^{ij}(x)$ satisfy a strong elliptic condition: for some $0 < \lambda \leq \Lambda$ we have

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in U \text{ and } \xi \in \mathbb{R}^n.$$

Exercise 1

Let the elliptic operator to be $Lv = -a^{ij}D_{ij}v$.
A continuous function $u : U \to \mathbb{R}$ is called sub-solution of L if for every ball $B \subset\subset U$ and for every function $h : B \to \mathbb{R}$ with $u \leq h$ on ∂B and $Lh = 0$ on B, we also have $u \leq h$ on B.
Show that in case $u \in C^2(U)$ this is equivalent to $Lu \leq 0$.
Hint: you may use that $Lh = 0$ in B with $h = v$ on ∂B can be solved for any $B \subset\subset U$ and $v \in W^{1,2}(U)$.

Exercise 2

Let u be a smooth solution of $Lu := -a^{ij}D_{ij}u = 0$ on U. Set $v := |Du|^2 + \mu u^2$. Show that

$$Lv \leq 0 \text{ in } U, \text{ if } \mu \text{ is large enough.}$$

Deduce that for some $C > 0$, depending only on a^{ij}, that

$$\|Du\|_{L^\infty(U)} \leq C\left(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}\right).$$

Exercise 3

Assume U is connected and has smooth boundary. Let u be a smooth solution of the Neumann-boundary-value problem

$$\begin{aligned} -\Delta u &= 0 \text{ in } U \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U. \end{aligned} \tag{1}$$

1. Show that $u \equiv C$ for some constant C using energy methods.
2. Show the same using the maximum principle.

Exercise 4

A function $u \in W_0^{2,2}(U)$ is a weak solution for this boundary-value problem for the *biharmonic equation*

$$\begin{aligned} \Delta^2 u &= f \text{ in } U \\ u = \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \end{aligned} \tag{2}$$

provided

$$\int_U \Delta u \Delta v = \int_U f v \, dx \text{ for all } v \in W_0^{2,2}(U).$$

Given $f \in L^2(U)$, show that there exists a unique weak solution of (2).
Hint: Show that for $u, v \in C^\infty(U)$ one has $\int_U \Delta u \Delta v - D_{ij} u D_{ij} v = \int_{\partial U} \frac{\partial u}{\partial \nu} \Delta v - D_i u \frac{\partial}{\partial \nu} D_i v$ i.e. for $u, v \in C_c^\infty(U)$ one has $\int_U \Delta u \Delta v = \int_U D_{ij} u D_{ij} v$. (We are using the Einstein sum convention: we sum over indices that appear twice). Conclude that an appropriate bilinear form fulfils the Lax-Milgram condition.