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Boundary and Eigenvalue Problems

11. exercise sheet

Exercise 1

Let $s_1 \leq s_2 \leq s_3$, $\frac{1}{p} + \frac{1}{q} = 1$ then the following interpolation holds

$$\|f\|_{H^{s_2}} \leq \|f\|_{H^{s_1}}^{\frac{1}{p}} \|f\|_{H^{s_3}}^{\frac{1}{q}} \text{ for all } f \in H^{s_1} \cap H^{s_3} \text{ and } \frac{1}{p}s_1 + \frac{1}{q}s_3 = s_2.$$

Conclude that for all $\epsilon > 0$, $s_2 \geq 1$ there is a constant $C_s > 0$ such that

$$\|f\|_{H^{s-1}} \leq \epsilon \|f\|_{H^s} + C_s \epsilon^{s-1} \|f\|_{L^2}.$$

Exercise 2

Let $L := -\sum_{i,j=1}^n D_j(a^{ij} D_i u)$ be a symmetric linear operator on $H_0^1(U)$ on a bounded domain U. As usual we assume that the coefficients are measurable and satisfy the ellipticity condition for some $0 < \lambda \leq \Lambda < \infty$

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in U \text{ and } \xi \in \mathbb{R}^n.$$

We denote with $a(u,v)$ the associated bilinear form i.e. $a(u,v) = \langle u, Lv\rangle_{L^2}$. Show that

$$\lambda_k := \max_{S \in \Sigma_{k-1}} \inf\{a(u,u) \colon u \in S^\perp, \|u\|_{L^2(U)} = 1\}.$$

Here Σ_{k-1} denotes the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$ and λ_k is the kth eigenvalue of L

On this sheet we assume that all coefficient are smooth, uniformly bounded functions on a domain $U \subset \mathbb{R}^n$ with smooth boundary. Furthermore we assume that the coefficients $a^{ij} = a^{ij}(x)$ satisfy a strong elliptic condition: for some $0 < \lambda \leq \Lambda$ we have

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in U \text{ and } \xi \in \mathbb{R}^n.$$

Exercise 3

Now we want to consider a non-symmetric linear operator:

$$Lu = -a^{ij} D_{ij} u + b^i D_i u + cu \text{ for } u \in C^2(U)$$

As usual we assume all coefficients to be in $C^\infty(\overline{U})$, $U \subset \mathbb{R}^n$ bounded, with smooth boundary, and a^{ij} satisfies the ellipticity condition as in the previous exercise. Moreover assume that $c \geq 0$. In this case the following theorem holds:

Theorem 1. *There exists a real eigenvalue λ_1 for the operator L, taken with zero boundary conditions, such that for any other eigenvalue $\lambda \in \mathbb{C}$ of L we have $\Re(\lambda) \geq \lambda_1$.*
Furthermore there exists a corresponding eigenfunction $\varphi_1 \in C^2(\overline{U})$ which is positive within U.

Prove the "max-min"representation formula:

$$\lambda_1 = \sup\left\{\inf_{x \in U} \frac{Lu(x)}{u(x)} \colon \colon u \in C^2(\overline{U}), u > 0 \text{ in } U \text{ and } u = 0 \text{ on } \partial U\right\}.$$

Hint: Consider the eigenfunction φ_1^* corresponding to λ_1 for the adjoint operator L^*.