07.07.2016
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Boundary and Eigenvalue Problems

12. exercise sheet

Exercise 1

Let $\Delta_i^h u(x) := h^{-1}(u(x+he_i) - u(x))$ be the difference quotient introduced on sheet 10. Show the following identities

1. if $u \in L^p(U)$ and $v \in L^q(U)$ then for any $U' \subset\subset U$, $h < \operatorname{dist}(U', \partial U)$ one has $\Delta_i^h(uv) \in L^r(U')$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and
$$\Delta_i^h(uv)(x) = \Delta_i^h u(x)\, v(x+he_i) + u(x)\,\Delta_i^h v(x)$$
2. if $u \in L^p(U)$, $U' \subset\subset U$, $h < \operatorname{dist}(U', \partial U)$ one has for any $\varphi \in C_c^1(U')$ that
$$\int_{U'} \Delta_h^i u\varphi = -\int_{U'} u\Delta_{-h}^i \varphi$$
3. Give an example of a function $u \in L^1(U), U \subset \mathbb{R}$ with the property that there is a constant $K > 0$ such that $\left\|\Delta^h u\right\|_{L^1(U')} < K$ for all $h > 0$, $U' \subset\subset U$, $h < \operatorname{dist}(U', \partial U)$ but $u \notin W^{1,1}(U)$.

Exercise 2

For $u \in C^2(S^{n-1}, \mathbb{R})$ we define the energy
$$E(u) := \int_{S^{n-1}} |\nabla u|^2 \, dvol_{S^{n-1}}.$$
A map $u \in C^2(S^{n-1}$ is called harmonic on S^{n-1} if $\Delta_{S^{n-1}} u = 0$, where $\Delta_{S^{n-1}}$ denotes the Laplace-Beltrami operator on S^{n-1}.

1. Show that $u \in C^2(S^{n-1})$ is harmonic if and only if it is a critical point of E.
2. Show that any harmonic map must be constant.
   a) using the maximum principle;
   b) using energy methods.

Exercise 3

The aim is to proof a part of the Sobolev embedding theorem for $p > n$:
$$W^{1,p}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n),$$
and there is a constant $C > 0$ s.t.
$$\|u\|_\infty \le C \, \|u\|_p^{1-\frac{n}{p}} \, \|Du\|_p^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\mathbb{R}^n). \tag{1}$$

1. Show that there are constants $A, B > 0$ s.t.

$$\|u\|_\infty \leq A\,\|u\|_p + B\,\|Du\|_p \text{ for all } u \in C_c^\infty. \tag{2}$$

   To do so use Green's representation formula:

$$f(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\Delta f(y)\,dy \text{ for all } f \in C_c^2(\mathbb{R}^n)$$

   where $\Gamma(x)$ is the Newton potential i.e. $\Gamma(x) = \frac{1}{2\pi}\ln(|x|)$ for $n = 2$ and $\Gamma(x) = \frac{1}{w_n n(2-n)}|x|^{2-n}$ for $n > 2$.
   Apply the above identity to $f(y) := \psi(x-y)g(y)$ for $g \in C_c^\infty$ arbitrary and a fixed cut-off function ψ. ($0 \leq \psi \leq 1$, $supp(\psi)$ is compact and $\psi \equiv 1$ in a neighbourhood of 0.). Integrate by parts and apply Hölders inequality. Use this to conclude (2).
2. Apply (2) to $u_\lambda(x) := u(\lambda x)$ for $u \in C_c^\infty$ and minimise in $\lambda > 0$ to conclude (1).