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Boundary and Eigenvalue Problems

2. exercises sheet

Exercise 1

Consider the periodic function

$$p(x) := \begin{cases} 1 & \text{if } 2m \le x < 2m+1 \text{ for some } m \in \mathbb{Z} \\ -1 & \text{if } 2m+1 \le x < 2m \text{ for some } m \in \mathbb{Z} \end{cases}$$

and the sequence $p_k(x) := p(kx)$.

1. Show that $p_k \in L^p(U)$ for all p and $U \subset \mathbb{R}$ bounded.
2. Show that p_k converges weakly to $q(x) = 0$ in $L^p(U)$.
3. Does $\{p_k^2\}_{k\in\mathbb{N}}$ contain a weakly convergent subsequence and does it converge to q^2 in $L^p(U)$?

Exercise 2

1. Show that if $u \in W^{1,p}(\mathbb{R}^n)$ with $n > p$, then $\frac{u}{r} \in L^p(\mathbb{R}^n)$, $r^2 = |x|^2$ and

$$\left\| \frac{u}{r} \right\|_{L^p(\mathbb{R}^n)} \le \frac{p}{n-p} \|Du\|_{L^p(\mathbb{R}^n)} .$$

   Hint : Prove the inequality for all $u \in C_c^\infty(\mathbb{R}^n)$ and then extend it to $W^{1,p}(\mathbb{R}^n)$. To prove it for $u \in C_c^\infty(\mathbb{R}^n)$ consider

$$\int_{\mathbb{R}^n} D_i\left(\frac{x_i}{|x|^p}\right) |u|^p$$

   and integrate by parts.
2. Show that the space of functions in $C_c^\infty(\mathbb{R}^n)$ which are 0 in a small ball around 0 is dense in $W^{1,p}(\mathbb{R}^n)$ for $n > p$.
   Hint: Use part 1. and an appropriate sequence of cut-off functions.

Exercise 3 ("Hydrogen Atom" Part I)

We want to study a mathematical model of the Hydrogen atom. The space of admissible wavefunctions and the energy is defined to be

$$H := \{\varphi \in W^{1,2}(\mathbb{R}^3, \mathbb{C}) \colon \|\varphi\|_{L^2(\mathbb{R}^3)} = 1\} \text{ and } E(\varphi) := \int_{\mathbb{R}^3} |D\varphi|^2 - Q\frac{|\varphi|^2}{r}\, dx \quad Q > 0.$$

1. Show that for any $\varphi \in W^{1,2}(\mathbb{R}^n)$ the energy is well defined i.e. $|E(\varphi)| < \infty$ and that is bounded below on H.
2. Show that the Energy is coercive on H i.e. show that $E(\varphi_n) \to \infty$ if $\varphi_n \in H$ with $\|D\varphi_n\|_{L^2(\mathbb{R}^3)} \to \infty$.
3. Show that $\inf_{\varphi\in H} E(\varphi) < 0$.
   Hint: Use exercise 2 for part 1. and 2. and consider the scaling $\varphi_\lambda(x) := \lambda^{\frac{3}{2}}\varphi(\lambda x)$ to conclude part 3.

The following exercise is optional. It's result is considered to be known, if not the exercise is thought as an option to repeat it.

Exercise 4 (Riesz Representation Theorem)

The aim of this Exercise is to show that $\mathcal{H}^* \equiv \mathcal{H}$ for any Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$.

1. Let $C \subset \mathcal{H}$ a closed convex subset. Show that to any $x \in \mathcal{H}$ there exists a unique $y \in C$ with
$$\|x - y\| = \inf_{z \in C} \|x - z\| .$$
Furthermore deduce that if C is a linear subspace then $x - y \in C^{\perp}$.
2. Show that for any $y \in \mathcal{H}$ the map
$$l(x) := (y, x)$$
defines an element in $\mathcal{H}^*$ and $\|l\| = \|y\|$.
3. Given $F \in \mathcal{H}^*$. Show that there exists $f \in \mathcal{H}$ with $F(x) = (y, x)$ for all $x \in \mathcal{H}$ and $\|F\| = \|f\|$.
Hint: Consider the linear subspace $C := \{x \in \mathcal{H} \colon F(x) = 0\}$. If $C \neq \mathcal{H}$ take $x_0 \notin C$ and let $y_0 \in C$ be the element of Question 1. Set $z = x_0 - y_0$ and consider the element $x - \frac{F(x)}{F(z)} z$ for any $x \in \mathcal{H}$ and evaluate F. Deduce that $f = \frac{F(z)}{\|z\|} z$ has the desired properties.