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Boundary and Eigenvalue Problems

3. exercises sheet

Exercise 1

Given $1 \leq p \leq \infty$ show that there is a constant $C > 0$ such that

$$\left\| f(x) - \fint_{B_1} f \right\|_{L^p(B_1)} \leq C \, \|Df\|_{L^p(B_1)} \text{ for any } f \in W^{1,p}$$

with $\fint_{B_1} f := \frac{1}{|B_1|} \int_{B_1} f \, dx$.
Hint: Prove it by contradiction. You may assume without a proof that $W^{1,p}(B_1)$ embeds compact in $L^p(B_1)$.

Exercise 2 ("Hydrogen Atom" Part II)

We want to prove the existence of a function $\varphi_0 \in H$ with

$$E(\varphi_0) = \inf_{\psi \in H} E(\psi) =: E_0, \tag{1}$$

with H and E had been defined on sheet 2:

$$H := \{\varphi \in W^{1,2}(\mathbb{R}^3, \mathbb{C}) \colon \|\varphi\|_{L^2(\mathbb{R}^3)} = 1\} \text{ and } E(\varphi) := \int_{\mathbb{R}^3} |D\varphi|^2 - Q\frac{|\varphi|^2}{r} \, dx \; Q > 0.$$

Let $\{\varphi_m\}_{m\in\mathbb{N}} \subset H$ be a minimizing sequence i.e.

$$\lim_{m\to\infty} E(\varphi_m) = E_0.$$

1. Show that $\{\varphi_m\}$ is uniformly bounded in $W^{1,2}(\mathbb{R}^3)$.
2. Argue why we can find a subsequence (after relabelling) $\{\varphi_m\}_m$ with the properties:
   - φ_m converges weakly in $W^{1,2}(\mathbb{R}^3)$
   - φ_m converges strongly in $L^2(U)$ for any bounded set $U \subset \mathbb{R}^3$

   to some element $\varphi_0 \in W^{1,2}(\mathbb{R}^3)$.
3. Show that
   $$\lim_{m\to\infty} \int_{\mathbb{R}^3} \frac{|\varphi_m|^2}{r} \, dx = \int_{\mathbb{R}^3} \frac{|\varphi|^2}{r} \, dx.$$
   Hint: use exercise 2 of sheet 2.
4. Conclude that $\liminf_{m\to\infty} E(\varphi_m) \geq E(\varphi_0)$.
5. Deduce from (1) that $\|\varphi_0\|_{L^2(\mathbb{R}^3)} = 1$ and hence $\varphi_0 \in H$ satisfying $E(\varphi_0) = E_0$.
6. Calculate the Euler-Lagrange equation to
   $$E(\varphi_0) = \inf_{\psi\in H} E(\psi) = \inf_{\substack{\varphi \in W^{1,2}(\mathbb{R}^3) \\ \varphi \neq 0}} \frac{E(\varphi)}{\|\varphi\|^2_{L^2(\mathbb{R}^3)}}.$$
   Deduce that φ_0 is an eigenfunction to the operator $-\Delta - \frac{Q}{r}$.

Exercise 3 ("chain rule and consequences")

1. Let $F \in C^1(\mathbb{R},\mathbb{R})$ with $F' \in L^\infty(\mathbb{R})$, $F(0) = 0$ and $U \subset \mathbb{R}^n$ open. Show that $F \circ f \in W^{1,p}(U)$ for any $f \in W^{1,p}(U)$ with

$$\partial_i(F \circ f)(x) = F'(f)\partial_i f(x) \text{ a.e. } .$$

(Bonus-Question)

2. Show that if $f \in W^{1,p}(U)$ then $f^+(x) := \max\{f(x), 0\}, f^-(x) := \max\{-f(x), 0\}$ and $|f| \in W^{1,p}(U)$ with

$$Df^+ = \begin{cases} Df & \text{a.e. on } \{f > 0\} \\ 0 & \text{a.e. on } \{f \le 0\} \end{cases}$$

$$Df^- = \begin{cases} 0 & \text{a.e. on } \{f \ge 0\} \\ -Df & \text{a.e. on } \{f < 0\} \end{cases}$$

$$D|f| = \begin{cases} Df & \text{a.e. on } \{f > 0\} \\ 0 & \text{a.e. on } \{f = 0\} \\ -Df & \text{a.e. on } \{f < 0\} \end{cases}$$

Hint: Consider the function

$$F_\epsilon(t) = \begin{cases} (t^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

Apply the previous part and let $\epsilon \to 0$.