Boundary and Eigenvalue Problems

3. exercises sheet

Exercise 1

Given $1 \leq p \leq \infty$ show that there is a constant $C > 0$ such that

$$
\left\| f(x) - \frac{1}{|B_1|} \int_{B_1} f \right\|_{L^p(B_1)} \leq C \|Df\|_{L^p(B_1)}
$$

for any $f \in W^{1,p}$ with

$$
|B_1| := \frac{1}{|B_1|} \int_{B_1} f \, dx.
$$

Hint: Prove it by contradiction. You may assume without a proof that $W^{1,p}(B_1)$ embeds compact in $L^p(B_1)$.

Exercise 2 ("Hydrogen Atom" Part II)

We want to prove the existence of a function $\varphi_0 \in H$ with

$$
E(\varphi_0) = \inf_{\psi \in H} E(\psi) =: E_0, \quad (1)
$$

with $H$ and $E$ had been defined on sheet 2:

$$
H := \{ \varphi \in W^{1,2}(\mathbb{R}^3, \mathbb{C}) : \|\varphi\|_{L^2(\mathbb{R}^3)} = 1 \} \text{ and } E(\varphi) := \int_{\mathbb{R}^3} |D\varphi|^2 - Q \frac{\|\varphi\|^2}{r} \, dx \, Q > 0.
$$

Let $\{\varphi_m\}_{m \in \mathbb{N}} \subset H$ be a minimizing sequence i.e.

$$
\lim_{m \to \infty} E(\varphi_m) = E_0.
$$

1. Show that $\{\varphi_m\}$ is uniformly bounded in $W^{1,2}(\mathbb{R}^3)$.
2. Argue why we can find a subsequence (after relabelling) $\{\varphi_m\}_m$ with the properties:
   - $\varphi_m$ converges weakly in $W^{1,2}(\mathbb{R}^3)$
   - $\varphi_m$ converges strongly in $L^2(U)$ for any bounded set $U \subset \mathbb{R}^3$
   to some element $\varphi_0 \in W^{1,2}(\mathbb{R}^3)$.
3. Show that

$$
\lim_{m \to \infty} \int_{\mathbb{R}^3} \frac{|\varphi_m|^2}{r} \, dx = \int_{\mathbb{R}^3} \frac{|\varphi|^2}{r} \, dx.
$$

Hint: use exercise 2 of sheet 2.
4. Conclude that $\lim \inf_{m \to \infty} E(\varphi_m) \geq E(\varphi_0)$.
5. Deduce from (1) that $\|\varphi_0\|_{L^2(\mathbb{R}^3)} = 1$ and hence $\varphi_0 \in H$ satisfying $E(\varphi_0) = E_0$.
6. Calculate the Euler-Lagrange equation to

$$
E(\varphi_0) = \inf_{\psi \in H} E(\psi) = \inf_{\varphi \in W^{1,2}(\mathbb{R}^3)} \frac{E(\varphi)}{\|\varphi\|^2_{L^2(\mathbb{R}^3)}}.
$$

Deduce that $\varphi_0$ is an eigenfunction to the operator $-\Delta - \frac{Q}{r}$.
Exercise 3 ("chain rule and consequences")

1. Let $F \in C^1(\mathbb{R}, \mathbb{R})$ with $F' \in L^\infty(\mathbb{R})$, $F(0) = 0$ and $U \subset \mathbb{R}^n$ open. Show that $F \circ f \in W^{1,p}(U)$ for any $f \in W^{1,p}(U)$ with

$$\partial_i (F \circ f)(x) = F'(f) \partial_i f(x) \text{ a.e.}$$

(Bonus-Question)

2. Show that if $f \in W^{1,p}(U)$ then $f^+(x) := \max\{f(x), 0\}, f^-(x) := \max\{-f(x), 0\}$ and $|f| \in W^{1,p}(U)$ with

$$Df^+ = \begin{cases} Df & \text{a.e. on } \{f > 0\} \\ 0 & \text{a.e. on } \{f \leq 0\} \end{cases}$$

$$Df^- = \begin{cases} 0 & \text{a.e. on } \{f \geq 0\} \\ -Df & \text{a.e. on } \{f < 0\} \end{cases}$$

$$D|f| = \begin{cases} Df & \text{a.e. on } \{f > 0\} \\ 0 & \text{a.e. on } \{f = 0\} \\ -Df & \text{a.e. on } \{f < 0\} \end{cases}$$

Hint: Consider the function

$$F_\epsilon(t) = \begin{cases} (t^2 + \epsilon^2)^{\frac{1}{2}} - \epsilon & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Apply the previous part and let $\epsilon \to 0$.  

2