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Boundary and Eigenvalue Problems

4. exercises sheet

Exercise 1

The aim of this exercise is to prove an extension theorem for $W^{1,p}(\mathbb{R}^n_+)$, $1 \leq p \leq \infty$ and $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n \colon x_n > 0\}$. In fact we want to show that for $u \in W^{1,p}(\mathbb{R}^n_+)$

$$\hat{u}(x', x_n) := \begin{cases} u(x', x_n) & \text{for } x_n > 0 \\ u(x', -x_n) & \text{for } x_n < 0 \end{cases}$$

defines an element in $W^{1,p}(\mathbb{R}^n)$ with $\|\hat{u}\|_{W^{1,p}(\mathbb{R}^n)} = 2^{\frac{1}{p}} \|u\|_{W^{1,p}(\mathbb{R}^n)}$. To do so proceed as follows:

1. Show that $\hat{u}$ defines an admissible extension with the claimed properties for $u \in C^1(\overline{\mathbb{R}^n_+}) \cap W^{1,p}(\mathbb{R}^n_+)$.
2. As introduced on the first exercise sheet let $\tau_y \circ f(x) := f(x + y)$ for any $y \in \mathbb{R}^n$ f measurable. Given an arbitrary $u \in W^{1,p}(\mathbb{R}^n_+)$ argue why
$$v_\epsilon(x) := \eta_\epsilon * (\tau_{\epsilon e_n} \circ u)(x)$$
is a well defined element in $W^{1,p}(\mathbb{R}^n_+)$ with $v_\epsilon \to u$ in $W^{1,p}(\mathbb{R}^n_+)$.
3. Conclude that $\hat{u}$ is well defined for general $u \in W^{1,p}(\mathbb{R}^n_+)$ by considering $\hat{v_\epsilon}$.

Exercise 2

Let U be bounded, with a C^1 boundary. Prove that there does not exists a bounded linear operator

$$T : L^p(U) \to L^p(\partial U), \quad 1 \leq p < \infty$$

such that $Tu = u|_{\partial U}$ whenever $u \in C^0(\overline{U}) \cap L^p(U)$.

1. (Bonus) Show that the trace operator $T : W^{1,1}(\mathbb{R}^n_+) \to L^1(\mathbb{R}^{n-1})$ is surjective.

Exercise 3

Let $L \in C^1(\mathbb{R}^n, \mathbb{R})$ satisfying the following conditions

(a) $L(p) \geq \alpha |p|^q - \beta$ for some $\alpha > 0$, $\beta \geq 0$ and $1 \leq q \leq \infty$;

(b) $p \mapsto L(p)$ is convex.

Let $U \subset \mathbb{R}^n$ be an open bounded domain with C^1 boundary. Given any $v_0 \in W^{1,q}(U)$ set $X := \{u \in W^{1,q}(U) \colon u - v_0 \in W^{1,q}_0(U)\}$. Furthermore we define the energy

$$E(v) := \int_U L(Du)\, dx.$$

We want to show that

$$E_0 := \inf_{u \in X} E(u)$$

is attained.

1. Show that under the above assumptions every minimizing sequence in bounded in $W^{1,p}(U)$.
2. Show that the E is lower semicontinuous i.e. if $v_k \rightharpoonup v$ weakly in $W^{1,p}(U)$ then

$$E(v) \leq \liminf_{k\to\infty} E(v_k).$$

Hint: Use that for a C^1 convex function one has $L(a) - L(b) \geq DL(b)(a-b)$ for all $a, b \in \mathbb{R}^n$.

3. Deduce from the previous parts the existence of a minimizer $u_0 \in X$ i.e.

$$E(u_0) = E_0.$$

4. (Bonus) Show that if $p \mapsto L(p)$ is C^2 and satisfies $\sum_{i,j=1}^n \partial^2_{p_i,p_j} L(p)\xi_i\xi_j \geq \theta|\xi|^2$ for some $\theta > 0$, then the minimizer is unique.