Boundary and Eigenvalue Problems

4. exercises sheet

Exercise 1

The aim of this exercise is to prove an extension theorem for \( W^{1,p}(\mathbb{R}^n_+) \), \( 1 \leq p \leq \infty \) and \( \mathbb{R}^n_+ := \{ x = (x', x_n) \in \mathbb{R}^n : x_n > 0 \} \). In fact we want to show that for \( u \in W^{1,p}(\mathbb{R}^n_+) \)

\[
\hat{u}(x', x_n) := \begin{cases} 
    u(x', x_n) & \text{for } x_n > 0 \\
    u(x', -x_n) & \text{for } x_n < 0
\end{cases}
\]

defines an element in \( W^{1,p}(\mathbb{R}^n) \) with \( \| \hat{u} \|_{W^{1,p}(\mathbb{R}^n)} = 2^{\frac{1}{p}} \| u \|_{W^{1,p}(\mathbb{R}^n_+)} \). To do so proceed as follows:

1. Show that \( \hat{u} \) defines an admissible extension with the claimed properties for \( u \in C^1(\mathbb{R}^n_+) \cap W^{1,p}(\mathbb{R}^n_+) \).
2. As introduced on the first exercise sheet let \( \tau_y \circ f(x) := f(x+y) \) for any \( y \in \mathbb{R}^n \) f measurable. Given an arbitrary \( u \in W^{1,p}(\mathbb{R}^n_+) \) argue why

\[
v_\epsilon(x) := \eta_\epsilon \ast (\tau_{\epsilon e_n} \circ u)(x)
\]

is a well defined element in \( W^{1,p}(\mathbb{R}^n_+) \) with \( v_\epsilon \rightarrow u \) in \( W^{1,p}(\mathbb{R}^n_+) \).
3. Conclude that \( \hat{u} \) is well defined for general \( u \in W^{1,p}(\mathbb{R}^n_+) \) by considering \( \hat{v}_\epsilon \).

Exercise 2

Let \( U \) be bounded, with a \( C^1 \) boundary. Prove that there does not exists a bounded linear operator

\[
T : L^p(U) \rightarrow L^p(\partial U), \quad 1 \leq p < \infty
\]

such that \( Tu = u|_{\partial U} \) whenever \( u \in C^0(\overline{U}) \cap L^p(U) \).

1. (Bonus) Show that the trace operator \( T : W^{1,1}(\mathbb{R}^n_+) \rightarrow L^1(\mathbb{R}^{n-1}) \) is surjective.

Exercise 3

Let \( L \in C^1(\mathbb{R}^n, \mathbb{R}) \) satisfying the following conditions

(a) \( L(p) \geq \alpha |p|^q - \beta \) for some \( \alpha > 0, \beta \geq 0 \) and \( 1 \leq q \leq \infty \);

(b) \( p \mapsto L(p) \) is convex.

Let \( U \subset \mathbb{R}^n \) be an open bounded domain with \( C^1 \) boundary. Given any \( v_0 \in W^{1,q}(U) \) set \( X := \{ u \in W^{1,q}(U) : u - v_0 \in W^{1,q}_0(U) \} \). Furthermore we define the energy

\[
E(v) := \int_U L(Du) \, dx.
\]

We want to show that

\[
E_0 := \inf_{u \in X} E(u)
\]

is attained.
1. Show that under the above assumptions every minimizing sequence in bounded in $W^{1,p}(U)$.

2. Show that the $E$ is lower semicontinuous i.e. if $v_k \rightharpoonup v$ weakly in $W^{1,p}(U)$ then

$$E(v) \leq \liminf_{k \to \infty} E(v_k).$$

*Hint:* Use that for a $C^1$ convex function one has $L(a) - L(b) \geq DL(b)(a - b)$ for all $a, b \in \mathbb{R}^n$.

3. Deduce from the previous parts the existence of a minimizer $u_0 \in X$ i.e.

$$E(u_0) = E_0.$$  

4. (Bonus) Show that if $p \mapsto L(p)$ is $C^2$ and satisfies $\sum_{i,j=1}^n \partial^2_{p_i,p_j} L(p) \xi_i \xi_j \geq \theta |\xi|^2$ for some $\theta > 0$, then the minimizer is unique.