9.05.2015
Dr. Ioannis Anapolitanos
Dr. Jonas Hirsch

Karlsruhe Institute of Technology

Boundary and Eigenvalue Problems

5. exercises sheet

The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is defined to be

$$\hat{f}(p) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ipx}\, dx.$$

The inverse Fourier transform is defined to be

$$\check{f}(p) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{ipx}\, dx.$$

Furthermore we define the inner product as

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) \bar{g}(x)\, dx.$$

Exercise 1

Verify the following properties of the Fourier transform:

1. For $f \in L^1$, $\hat{f} \in C^0(\mathbb{R}^n)$ with $\left\|\hat{f}\right\|_{L^\infty} \le (2\pi)^{-\frac{n}{2}} \|f\|_{L^1}$.
2. For $f, g \in L^1 \cap L^2$ one has $\widehat{f * g} = (2\pi)^{\frac{n}{2}} \hat{f}\, \hat{g}$.
3. For $f, g \in L^1$ one has $\langle \check{f}, g \rangle = \langle f, \hat{g} \rangle$.

Exercise 2

1. Calculate the Fourier transform of the Gaussian
$$G(x) := e^{-\frac{|x|^2}{2}}.$$
2. Show that for $f \in L^1$ and $\rho > 0$ one has $\widehat{\rho^{-n} f\left(\frac{x}{\rho}\right)} = \hat{f}(\rho x)$.
3. For $\rho > 0$ define $G_\rho(x) := \rho^{-n} G\left(\frac{x}{\rho}\right)$. Proof that for $f \in L^p$ one has in the L^p sense
$$\lim_{\rho \to 0} (G_\rho * f)(x) = f(x) \left(\int_{\mathbb{R}^n} G(y) dy \right) = (2\pi)^{\frac{n}{2}} f(x).$$

Exercise 3

1. Prove the Plancherel formula:
For $f \in L^1 \cap L^2$ we have $\|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2} = \|\check{f}\|_{L^2}$.
Hint: Consider the integral $\int_{\mathbb{R}^n} |\hat{f}|^2(x) G(\rho x)\, dx$, use the previous exercises and take the limit $\rho \to 0$. Furthermore observe that $\hat{\bar{f}} = \bar{\check{f}}$.

2. Prove the Fourier inversion:

$$\check{\hat{f}}(x) = f(x) \text{ for a.e. } x \in \mathbb{R}^3 \text{ and } f \in L^1 \cap L^2.$$

Hint : Combine the Plancherel formula with Exercise 1.

Exercise 4

Show that there is no function $f \in L^2$ that is an eigenfunction of the Laplace operator i.e. $-\Delta f = \lambda f$ for some $\lambda \geq 0$ and $f \in L^2$ implies $f = 0$.
Hint: You may use the Fourier transform.

Exercise 5

We want to prove the Heisenberg uncertainty principle:
Let $\varphi \in H^2(\mathbb{R}^3, \mathbb{C})$ be a wave function of a particle i.e. $\|\varphi\|_{L^2} = 1$. The wave function has the following physical interpretations:

- $|\varphi|^2(x)$ is the probability density that the particle is at x.

$$\bar{x}_i := \int_{\mathbb{R}^3} x_i |\varphi|^2 \, dx$$

  the expected position in the ith coordinate with variance

$$\sigma(x_i)^2 := \int_{\mathbb{R}^3} (x_i - \bar{x}_i)^2 |\varphi|^2 \, dx;$$

- $|\hat{\varphi}|^2(x)$ is the probability density that the particle has momentum p.

$$\bar{p}_i := \int_{\mathbb{R}^3} p_i |\hat{\varphi}|^2 \, dp$$

  the expected momentum in the ith direction with variance

$$\sigma(p_i)^2 := \int_{\mathbb{R}^3} (p_i - \bar{p}_i)^2 |\varphi|^2 \, dp.$$

Proof Heisenberg uncertainty principle:

$$\frac{1}{2} \leq \sigma(x_i)\sigma(p_i).$$