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Boundary and Eigenvalue Problems

6. exercise sheet

Exercise 1

Assume that $U \subset \mathbb{R}^n$ is a bounded connected domain. A function $u \in H^1(U)$ is a weak solution of *Neumann's problem*

$$\begin{aligned} -\Delta u &= f \text{ in } U \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \end{aligned} \tag{1}$$

if

$$\int_U Du \cdot Dv \, dx = \int_U fv \, dx \quad \text{for all } v \in H^1(U). \tag{2}$$

1. Suppose that $u \in C^2(\overline{U})$ and $f \in C^0(\overline{U})$. Check that in this case (1) is indeed equivalent to (2).
2. Suppose that $f \in L^2(U)$. Show that (1) has a weak solution if and only if

$$\int_U f \, dx = 0.$$

Exercise 2

1. Given $1 \leq p \leq \infty$ and $U \subset \mathbb{R}^n$ bounded with C^1-regular boundary show that there is a constant $C = C(U) > 0$ such that

$$\left\| f(x) - \fint_{\partial U} Tf \right\|_{L^p(U)} \leq C \left\| Df \right\|_{L^p(U)} \text{ for any } f \in W^{1,p}(U)$$

   with $\fint_{\partial U} Tf := \frac{1}{|\partial U|} \int_{\partial U} Tf \, dH^{n-1}$ and Tf is the trace of f.

2. Let $U \subset \mathbb{R}^n$ with $|U| < \infty$. Show that Poincarés inequality does not hold on $W^{1,p}(U)$ i.e. there cannot exists a constant such that

$$\|f\|_{L^p(U)} \leq C \|Df\|_{L^p(U)} \text{ for any } f \in W^{1,p}(U).$$

3. Let $U \subset \mathbb{R}^n$ such that U contains balls of arbitrary large radius, then Poincarés inequality cannot hold on U for $W_0^{1,p}(U)$ either.

Exercise 3

Let X be a real Hilbertspace with dual X'.

1. Show that any bilinear continuous form a on $X \times X$ defines a continuous linear operator $A : X \to X'$.
2. Show that the reverse holds true as well i.e. any continuous linear operator $A : X \to X'$ defines a continuous bilinear form.

3. Show the reverse of the Lax-Milgram lemma: Suppose $A : X \to X'$ is a continuous linear operator that is injective and has close range. Then there exists $\alpha > 0$ such that $\|Av\|_* \geq \alpha \|v\|$ for all $v \in X$.

The following exercise is a "Bonus-Question".

Exercise 4 (traces on H^s)

Let $s > \frac{1}{2}$. Show that there is a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ such that $Tf(x') = f(x', 0)$ for all $f \in C_c^\infty(\mathbb{R}^n)$.
Hint: Let $f \in C_c^\infty(\mathbb{R}^n)$ and $g(x') := f(x', 0)$. Show that $\hat{g}(p') = \int_{\mathbb{R}} \hat{f}(p', p_n)\, dp_n$ and try to estimate $\int_{\mathbb{R}} (1 + |p|^2)^{-\frac{s}{2}} (1 + |p|^2)^{\frac{s}{2}} \hat{f}(p', p_n)\, dp_n$.