Boundary and Eigenvalue Problems

8. exercise sheet

Exercise 1

Let $X$ be a Banachspace and $A : X \to X$ a linear continuous operator. Let $\lambda_1, \lambda_2, \ldots$ eigenvalues of $A$. Furthermore we denote with $E_{\lambda_i}$ the eigenspace of $A$ to the eigenvalue $\lambda_i$ i.e. $E_{\lambda_i} = \ker(A - \lambda i)$.

Show that any finite family of unite eigenvectors to different eigenvalues are linearly independent i.e. given $\{v_i\}_{i=1}^{k}$ with $\|v_i\| = 1, v_i \in E_{\lambda_{l_i}}$ for $i = 1, \ldots, k$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ then

$$\sum_{i=1}^{k} a^i v_i = 0$$

implies $a_i = 0$ for all $i$.

Exercise 2

Given $U \subset \mathbb{R}^n$ and $\epsilon > 0$. Set $U^\epsilon := \{x \in U : \text{dist}(x, U^c) \geq \epsilon\}$. Show that there exists $\varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R})$ with $0 \leq \varphi \leq 1, \varphi \equiv 1$ on $U^\epsilon$ and $\text{spt}(\varphi) \subset U^2$.

Exercise 3

Let $a^{ij} \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Furthermore let $u \in H^1(U)$ be a weak solution of

$$-\partial_i a^{ij} \partial_j u = f \text{ on } U.$$ 

Show that there exists $S \in GL(n)$ such that $v(x) := u(Sx)$ solves

$$-\Delta v = \tilde{f} \text{ for } \tilde{f}(x) = f(Sx).$$

Conclude that if $f$ is a smooth function, that $u$ is smooth in the interior of $U$. 