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Boundary and Eigenvalue Problems

8. exercise sheet

Exercise 1

Let X be a Banachspace and $A : X \to X$ a linear continuous operator. Let $\lambda_1, \lambda_2, \dots$ eigenvalues of A. Furthermore we denote with E_{λ_i} the eigenspace of A to the eigenvalue λ_i i.e. $E_{\lambda_i} = \ker(A - \lambda \mathbf{1})$.
Show that any finite family of unite eigenvectors to different eigenvalues are linearly independent i.e. given $\{v_i\}_{i=1}^k$ with $\|v_i\| = 1, v_i \in E_{\lambda_{l_i}}$ for $i = 1, \dots, k$ and $\lambda_{l_i} \neq \lambda_{l_j}$ for $i \neq j$ then

$$\sum_{i=1}^{k} a^i v_i = 0$$

implies $a_i = 0$ for all i.

Exercise 2

Given $U \subset \mathbb{R}^n$ and $\epsilon > 0$. Set $U^\epsilon := \{x \in U\colon\ \operatorname{dist}(x, U^c) \geq \epsilon\}$. Show that there exists $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on U^ϵ and $\operatorname{spt}(\varphi) \subset U^{\frac{\epsilon}{2}}$.

Exercise 3

Let $a^{ij} \in \mathbb{R}^{n \times n}$ be a symmetric positive definit matrix. Furthermore let $u \in H^1(U)$ be a weak solution of

$$-\partial_i a^{ij} \partial_j u = f \text{ on } U.$$

Show that there exists $S \in GL(n)$ such that $v(x) := u(Sx)$ solves

$$-\Delta v = \tilde{f} \text{ for } \tilde{f}(x) = f(Sx).$$

Conclude that if f is a smooth function, that u is smooth in the interior of U.