Boundary and Eigenvalue Problems

9. exercise sheet

Exercise 1

Consider the operator

\[ Lu := -\partial_i (a^{ij} \partial_j u) + b^i \partial_i u + cu \]

whose coefficients \( a^{ij}, c, b^i \) are assumed to be smooth functions on a bounded domain \( U \subset \mathbb{R}^n \) with smooth boundary. (We use the Einstein sum convention.) Furthermore we assume that \( c \) is non-negative. The coefficients \( a^{ij} \) satisfy a strict ellipticity conditions i.e. there exists \( \lambda > 0 \) such that

\[ a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \]

for all \( x \in U \) and \( \xi \in \mathbb{R}^n \).

Show that \( Lu = f \) can be solved for any \( f \in L^2 \) (or if you want \( f \in W^{1,2}_0(U) \)).

Exercise 2

We want to show that there is no maximum principle for operators of fourth order. Exemplary consider

\[ \Delta \Delta u = 0 \]

on \( B_1 \subset \mathbb{R}^n \).

1. Give an example of a function that attains a strict minimum and/or maximum in the interior but is not constant. (Which would be excluded in case of a maximum principle.)
2. Is the solution to \( \Delta \Delta u = 0 \) unique with \( u = 0 \) on \( \partial B_1 \)?

Exercise 3 (Difference quotient)

We define the difference quotient in direction \( e_i \) by

\[ \Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}, \quad h \neq 0. \]

Proof the following statements:

1. Let \( u \in W^{1,p}(U), 1 \leq p \leq \infty \). Then \( \Delta_i^h u(x) \in L^p(U') \) for any \( U' \subset \subset U \) satisfying \( h < \text{dist}(U', \partial U) \) and we have

\[ \| \Delta_i^h u \|_{L^p(U')} \leq \| D_i u \|_{L^p(U)}. \]

2. Let \( u \in L^p(U), 1 < p < \infty \) and suppose there exists a constant \( K \) such that \( \Delta_i^h u \in L^p(U') \) and \( \| \Delta_i^h u \|_{L^p(U')} \leq K \) for all \( h > 0 \) and \( U' \subset \subset U \) satisfying \( h < \text{dist}(U', \partial U) \). Then the weak derivative \( D_i u \) exists and satisfies \( \| D_i u \|_{L^p(U)} \leq K \).