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Boundary and Eigenvalue Problems

9. exercise sheet

Exercise 1

Consider the operator

$$Lu := -\partial_i \left(a^{ij}\partial_j u\right) + b^i\partial_i u + cu$$

whose coefficients a^{ij}, c, b_i are assumed to be smooth functions on a bounded domain $U \subset \mathbb{R}^n$ with smooth boundary. (We use the Einstein sum convention.) Furthermore we assume that c is non-negative. The coefficients a^{ij} satisfy a strict ellipticity conditions i.e. there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } x \in U \text{ and } \xi \in \mathbb{R}^n.$$

Show that $Lu = f$ can be solved for any $f \in L^2$ (or if you want $f \in W_0^{1,2}(U)^*$).

Exercise 2

We want to show that there is no maximum principle for operators of fourth order. Exemplary consider

$$\Delta\Delta u = 0 \text{ on } B_1 \subset \mathbb{R}^n.$$

1. Give an example of a function that attains a strict minimum and/or maximum in the interior but is not constant. (Which would be excluded in case of a maximum principle.)
2. Is the solution to $\Delta\Delta u = 0$ unique with $u = 0$ on ∂B_1?

Exercise 3 (Difference quotient)

We define the difference quotient in direction e_i by

$$\Delta_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}, \quad h \neq 0.$$

Proof the following statements:

1. Let $u \in W^{1,p}(U)$, $1 \leq p \leq \infty$. Then $\Delta_i^h u(x) \in L^p(U')$ for any $U' \subset\subset U$ satisfying $h < \operatorname{dist}(U', \partial U)$ and we have
$$\left\|\Delta_i^h u\right\|_{L^p(U')} \leq \|D_i u\|_{L^p(U)}.$$
2. Let $u \in L^p(U)$, $1 < p < \infty$ and suppose there exists a constant K such that $\Delta_i^h u \in L^p(U')$ and $\left\|\Delta_i^h u\right\|_{L^p(U')} \leq K$ for all $h > 0$ and $U' \subset\subset U$ satisfying $h < \operatorname{dist}(U', \partial U)$. Then the weak derivative $D_i u$ exists and satisfies $\|D_i u\|_{L^p(U)} \leq K$.