Fourier inversion formula: Let $f \in L^2(\mathbb{R}^n)$. Then (1)
$f = \check{\hat{f}}$. In particular, if $\hat{f} \in L^1(\mathbb{R}^n)$ then

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\xi}\hat{f}(\xi)\,d\xi. \quad (1)$$

Fractional Sobolev spaces in $\mathbb{R}^n$.

Def. Let $s \geq 0$. We define $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2(\mathbb{R}^n)\}$.
We equip $H^s(\mathbb{R}^n)$ with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \left(\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

Thm: For any $s \geq 0$ $(H^s(\mathbb{R}^n), \|\cdot\|_{H^s(\mathbb{R}^n)})$ is a Banach space.
The proof is based on the fact that H^s can be realised as an $L^2_\mu(\mathbb{R}^n)$ space on the Fourier transform where

$$d\mu = (1+|\xi|^2)^{\frac{s}{2}} d\xi.$$

Intuition: Recall that for $u \in C_c^\infty(\mathbb{R}^n)$
$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$. So requiring that $(1+|\xi|^2)^{\frac{s}{2}}\hat{u} \in L^2$ is like requiring that $|\xi|^s\hat{u} \in L^2$ or that u has "s" derivatives in L^2. The bigger the s the more the required decay of $\hat{u}$. From (1) one sees then intuitively that the bigger frequencies have small contribution and therefore u is in some sense expected to be smooth.

<u>Theorem</u>: Let $u \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Then $u \in H^k(\mathbb{R}^n)$ $\Longleftrightarrow$ $u \in W^{k,2}(\mathbb{R}^n)$. Moreover the norms $\|\cdot\|_{W^{k,2}(\mathbb{R}^n)}$, $\|\cdot\|_{H^k(\mathbb{R}^n)}$ are equivalent, i.e. $\exists\, C_1, C_2 > 0$ depending only on n and k such that.

$$C_1 \|u\|_{H^k(\mathbb{R}^n)} \leq \|u\|_{W^{k,2}(\mathbb{R}^n)} \leq C_2 \|u\|_{H^k(\mathbb{R}^n)} \quad \forall u \in H^k(\mathbb{R}^n).$$

<u>Proof</u>: "$\Longleftarrow$" Suppose that $u \in W^{k,2}(\mathbb{R}^n)$.

Claim: $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$ for almost all $\xi \in \mathbb{R}^n$.

To show the claim we first recall that it holds when $u \in C_c^\infty(\mathbb{R}^n)$. Otherwise we consider $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ with $u_n \to u$ on $W^{k,2}(\mathbb{R}^n)$.

Then $\widehat{D^\alpha u_n}(\xi) = (i\xi)^\alpha \widehat{u_n}(\xi) \quad \forall \xi \in \mathbb{R}^n$.

Since $u_n \to u$ in L^2 and $\widehat{D^\alpha u_n} \to \widehat{D^\alpha u}$ in L^2. it follows that $\widehat{u_n} \to \hat{u}$ in L^2 and $\widehat{D^\alpha u_n} \to \widehat{D^\alpha u}$ in L^2. Therefore we can extract a subsequence u_{n_k} of u_n s.t $\widehat{u_{n_k}} \to \hat{u}$ a.e. and $\widehat{D^\alpha u_{n_k}} \to \widehat{D^\alpha u}$ a.e. It follows that $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$ a.e. Therefore $\xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ $\forall |\alpha| \leq k$ and from this it easily follows that $(1+|\xi|^2)^{\frac{k}{2}} \hat{u} \in L^2(\mathbb{R}^n) \Longrightarrow u \in H^k(\mathbb{R}^n)$.

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"$\Longrightarrow$" Assume that $u \in H^k(\mathbb{R}^n)$. Then $\xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ when $|\alpha| \le k$. Therefore, if $v \in C_c^\infty(\mathbb{R}^n)$ then

$$(-1)^{|\alpha|} \int_{\mathbb{R}^n} u(x) \overline{D^\alpha v(x)}\, dx \overset{\text{Plancherel}}{=\!=\!=} (-1)^{|\alpha|} \int_{\mathbb{R}^n} \hat{u}(\xi) \overline{\widehat{D^\alpha v}(\xi)}\, d\xi$$

$$= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \hat{u}(\xi) \overline{(i\xi)^\alpha \hat{v}(\xi)}\, d\xi = \int_{\mathbb{R}^n} (i\xi)^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)}\, d\xi$$

$$\overset{\text{Plancherel}}{=\!=\!=} \int_{\mathbb{R}} \check{f}(x) \overline{v(x)}\, dx, \text{ where } f(\xi) = (i\xi)^\alpha \hat{u}(\xi) \in L^2.$$

It follows that $D^\alpha u \in L^2(\mathbb{R}^n)$ and $D^\alpha u = \check{f}$ or $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$ a.e. Since $|\alpha| \le k$ was arbitrary we obtain that $u \in W^{k,2}(\mathbb{R}^n)$. The equivalence of the norms relies on the fact that there exists $C_1, C_2 > 0$ such that for all $\xi \in \mathbb{R}^n$ we have that

$$C_1^2 (1+|\xi|^2)^k \le \sum_{|\alpha| \le k} |\xi^{2\alpha}| \le C_2^2 (1+|\xi|^2)^k.$$

__Remark:__ It is possible to define $H^s(\mathbb{R}^n)$ for $s<0$ but in this case one has to define $\hat{u}$ carefully.

Now we are going to introduce the first tool that we ~~will~~ use in order to prove regularity of (4) solutions.

<u>Thm</u> (A Sobolev imbedding theorem).

(i) If $s > \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Moreover $\exists\, C_{n,s} > 0$ depending on n and s such that $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C_{n,s} \|u\|_{H^s(\mathbb{R}^n)}$ $\forall\, u \in H^s(\mathbb{R}^n)$.

(ii) Let $d \in \mathbb{N}$. If $s > \frac{n}{2} + d$ then $H^s(\mathbb{R}^n) \subset C^d(\mathbb{R}^n)$. Moreover $\exists\, C > 0$ depending on n, s, d such that

$$\max_{|\alpha| \leq d} \|D^\alpha u\|_{L^\infty(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)} \quad \forall\, u \in H^s(\mathbb{R}^n).$$

<u>Proof</u>: (i). We will first show that $\hat{u} \in L^1$. Indeed,

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)|\, d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{s}{2}} (1+|\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)|\, d\xi$$

$$\overset{\text{Cauchy-Schwarz}}{\leq} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi \right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}}_{= \|u\|_{H^s}}.$$

Polar coordinates.

area of the unit sphere

$$|S^{n-1}| \int_0^\infty (1+r^2)^{-s} r^{n-1}\, dr.$$

But if $s > \frac{n}{2}$ then the last integral converges. Therefore.

$\int |\hat{u}(\xi)| d\xi \leq C_{n,s} \|u\|_{H^s(\mathbb{R}^n)}$, where

$C_{n,s} := \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi \right)^{\frac{1}{2}} < \infty$ (when $s > \frac{n}{2}$).

Since $\hat{u} \in L^1$ it follows that

$u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi} \hat{u}(\xi) d\xi \quad \forall x \in \mathbb{R}^n$. In particular

$$|u(x)| = \frac{1}{(2\pi)^{\frac{n}{2}}} \int |\hat{u}(\xi)| d\xi \leq \frac{C_{n,s}}{(2\pi)^{\frac{n}{2}}} \|u\|_{H^s(\mathbb{R}^n)}.$$

So $\|u\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_{n,s}}{(2\pi)^{\frac{n}{2}}} \|u\|_{H^s(\mathbb{R}^n)}$.

To prove continuity we observe that

$$u(x+h) - u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \underbrace{(e^{i(x+h)\xi} - e^{ix\xi}) \hat{u}(\xi)}_{\substack{\downarrow h \to 0 \\ 0}} d\xi.$$

(almost everywhere)

But then since $|e^{i(x+h)\xi} - e^{ix\xi}| |\hat{u}(\xi)| \leq 2|\hat{u}(\xi)|$ we can apply the dominated convergence theorem to show that $\lim_{h \to 0} (u(x+h) - u(x)) = 0$.
In fact since $|u(x+h) - u(x)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |e^{ih\xi} - 1| |\hat{u}(\xi)| d\xi$
~~the~~ it easily follows that u is

uniformly continuous.

(ii). If $|\alpha| \leq d$ then $D^\alpha u \in H^{s-d}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ * It follows that $D^\alpha u$ is continuous for all $|\alpha| \leq d$ so $u \in C^d(\mathbb{R}^n)$. The inequality follows easily. * because $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$ $\forall u \in H^d \subset H^s$

(we showed that we proving $u \in W^{k,2} \Rightarrow u \in H^k$)

The following theorem allows us to prove regularity (smoothness) of eigenfunctions of Schrödinger operators ($\cancel{-\Delta+V}$ the type $-\Delta + V$). After this theorem an application follows.

<u>Theorem</u>: Suppose that $u \in L^2(\mathbb{R}^n)$. Then $\forall s \geq 0$. $\Delta u \in H^s(\mathbb{R}^n)$ $\iff$ $u \in H^{s+2}(\mathbb{R}^n)$.

<u>Remark</u>: Things become ~~completely~~ different on $U \subseteq \mathbb{R}^n$. So for example if U is open and bounded and $u \in L^2(U)$, $\Delta u \in L^2(U)$ it does not follow that $u \in W^{2,2}(U)$. For this reason showing regularity of e.g. eigenfunctions of the Laplacian on bounded domains is much harder.

<u>Proof</u>: "$\Leftarrow$" If $u \in H^{s+2}(\mathbb{R}^n) \subset H^2(\mathbb{R}^n)$ then as we showed in the previous proof $\widehat{-\Delta u}(\xi) = |\xi|^2 \hat{u}(\xi)$.

But since $(1+|\xi|^2)^{\frac{s+2}{2}}\hat{u} \in L^2(\mathbb{R}^n)$ it follows then that $(1+|\xi|^2)^{\frac{s}{2}}\widehat{\Delta u} \in L^2(\mathbb{R}^n)$. 7

"$\Longrightarrow$" We will show the slightly weaker statement: if $u \in H^2(\mathbb{R}^n)$ and $\Delta u \in H^s(\mathbb{R}^n)$ then

$$u \in H^{s+2}(\mathbb{R}^n).$$

Indeed, since $u \in H^2(\mathbb{R}^n)$ we have $\widehat{-\Delta u}(\xi) = |\xi|^2\hat{u}(\xi)$. Since $(1+|\xi|^2)^{\frac{s}{2}}\widehat{-\Delta u}(\xi) = (1+|\xi|^2)^{\frac{s}{2}}|\xi|^2\hat{u} \in L^2(\mathbb{R}^n)$ and $\hat{u} \in L^2(\mathbb{R}^n)$ it follows that $(1+|\xi|^2)^{\frac{s+2}{2}}\hat{u} \in L^2(\mathbb{R}^n)$ as desired. $\square$

Remark: It is deciding that $|\widehat{D^\alpha u}(\xi)|$

$$= |\xi^\alpha \hat{u}(\xi)| \leq |\xi|^2|\hat{u}(\xi)| \quad \forall |\alpha| \leq 2$$

So if $\Delta u \in H^s(\mathbb{R}^n)$ then $D^\alpha u \in L^2(\mathbb{R}^n)$ $\forall |\alpha| = 2$. However, this argument does not work on domains $U \subset \mathbb{R}^n$.

"**Bonus**" Proof that if $u \in L^2(\mathbb{R}^n)$ and $\Delta u \in H^s(\mathbb{R}^n)$

Then $u \in H^{s+2}(\mathbb{R}^n)$. We defined the space S of Schwarz functions by

$S = \{ f \in C^\infty(\mathbb{R}^n) : \|x^\alpha D^\beta f\|_{L^\infty(\mathbb{R}^n)} < \infty$, for all multi-indices $\alpha, \beta \}$. One can show that.

$f \in S \iff \hat{f} \in S$. Moreover $\hat{\ } : S \to S$ in one to one and onto. Now for all $f \in H^s$ we have that

$$\int_{\mathbb{R}^n} \Delta u(x) f(x)\, dx = \int_{\mathbb{R}^n} u(x) \Delta f(x)\, dx = \int_{\mathbb{R}^n} \hat{u}(\xi) \widehat{\Delta f}(\xi) d\xi = -\int |\xi|^2 \hat{u}(\xi) \hat{f}(\xi) d\xi$$

|| Plancherel

$$\int_{\mathbb{R}^n} \widehat{\Delta u}(\xi) \hat{f}(\xi)\, d\xi.$$ It follows that

$$\int \left(\widehat{\Delta u}(\xi) + |\xi|^2 \hat{u}(\xi) \right) \hat{f}(\xi)\, d\xi = 0$$ or that

$$\int \left(\widehat{\Delta u}(\xi) + |\xi|^2 \hat{u}(\xi) \right) v(\xi)\, d\xi = 0 \quad \forall v \in S$$

Since $\widehat{\Delta u}(\xi) \in L^2$, $|\xi|^2 \hat{u} \in L^2_{loc}$ this implies that $\widehat{\Delta u}(\xi) = -|\xi|^2 \hat{u}(\xi)$. Now we can argue as in the previous proof.

Application 1: Suppose that $V:\mathbb{R}^3\to\mathbb{R}$ is C_c^∞ and let $\psi:\mathbb{R}^3\to\mathbb{C}$ with $\psi\in L^2(\mathbb{R}^3)$ be such that $(-\Delta+V)\psi=E\psi$ for some $E\in\mathbb{R}$.

Then $\psi\in C^\infty(\mathbb{R}^3)$.

Proof: Recall: if $V\in C_c^\infty$ and $\psi\in H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$ (here $k\in\mathbb{N}$) then $V\psi\in H^{k+2}(\mathbb{R}^n)$.

With this observation since

$$-\Delta\psi = V\psi+E\psi.$$ we argue as follows.

$$\psi\in L^2 \Longrightarrow (V\psi+E\psi)\in L^2 \Longrightarrow -\Delta\psi\in L^2=H^0 \xrightarrow[\text{Theorem}]{\text{Last}} \psi\in H^2.$$

$$\psi\in H^2 \Longrightarrow (V\psi+E\psi)\in H^2 \Longrightarrow -\Delta\psi\in H^2 \Longrightarrow \psi\in H^4.$$

Arguing like that we obtain that $\psi\in H^\infty(\mathbb{R}^3) := \bigcap_{s\geq 0} H^s(\mathbb{R}^3)$ and therefore by the Sobolev imbedding theorem $\psi\in C^\infty(\mathbb{R}^3)$. This kind of argument with which we obtained that $\psi\in H^\infty(\mathbb{R}^3)$ is called a bootstrapping argument. The same argument works if $V\in W^{k,\infty}(\mathbb{R}^3)$. $\forall k\in\mathbb{N}$ (see exercise sheet 1, exercise 3).