Example: Let $U \subseteq \mathbb{R}^n$ bounded and (1).
$X = \{ u \in W_0^{1,2}(U) : \|u\|_{L^2(U)} = 1 \}$. Show that there is a minimizer of $E[u] := \int_U |\nabla u|^2 dx$ in X.

Proof: Let $(u_n)_{n \in \mathbb{N}} \subset X$ be a minimizing sequence i.e $\int_U |\nabla u_n|^2 dx \to \inf_{u \in X} \int_U |\nabla u|^2 dx =: a$

Then $$\|u_n\|^2_{W_0^{1,2}(U)} = \underbrace{\int_U |u_n|^2 dx}_{=1} + \underbrace{\int_U |\nabla u_n|^2 dx}_{\downarrow \, a}$$

is bounded. Therefore, $\exists\, u_0 \in W_0^{1,2}(U)$ such that $u_n \rightharpoonup u_0$ in $W_0^{1,2}(U)$. Since $i : W_0^{1,2}(U) \to L^2(U)$ is compact by the Rellich-Kondrachov compactness theorem, it follows that $u_n \to u_0$ in $L^2(U)$ and in particular $\|u_0\|_{L^2(U)} = 1$ and as a consequence $u_0 \in X$. Moreover, since $u_n \rightharpoonup u_0$ in $W_0^{1,2}(U)$ we obtain $\|u_0\|_{W^{1,2}(U)} \leq \liminf_{n \to \infty} \|u_n\|_{W^{1,2}(U)}$

$$\overset{\|u_0\|_{L^2(U)} = \|u_n\|_{L^2(U)} = 1}{\Longrightarrow} \int_U |\nabla u_0|^2 dx \leq \lim_{n \to \infty} \int_U |\nabla u_n|^2 = \inf_{u \in X} \int_U |\nabla u|^2 dx.$$

Since $u_0 \in X$ it follows that u_0 is a minimizer.

Let $u_0 \in X$ be a minimizer of E in X. (2)

Then $-\Delta u_0 = E(u_0) u_0$.

Proof: We have that

$$\int_U |\nabla u|^2 dx - E(u_0) \int_U |u|^2 dx \geq 0 \quad \forall u \in W_0^{1,2}(U).$$

Therefore if $v \in W^{1,2}(U)$ then

$$f(s) := \int_U |\nabla(u_0+sv)|^2 - E(u_0) \int_U |u_0+sv|^2 dx \geq 0$$

with equality if $s=0$. Thus f has a minimum at 0 and therefore $f'(0)=0$.

since $$f(s) = \int_U |\nabla u_0|^2 dx + 2s\int_U \nabla u_0 \cdot \nabla v\, dx + s^2 \int_U |\nabla v|^2 dx$$
$$- E(u_0)\left(\int_U |u_0|^2 + 2s \int_U u_0 \cdot v\, dx + s^2 \int |v|^2 dx\right).$$

from $f'(0)=0$ it follows that

$$\int_U \nabla u_0 \nabla v\, dx = E(u_0) \int_U u_0 \cdot v\, dx.$$ So if $v \in C_c^\infty(U)$

then $$\int_U u_0 (-\Delta v) dx = E(u_0) \int_U u_0 v\, dx.$$

Since $v \in C_c^\infty(U)$ was arbitrary we obtain that in the sence of weak derivatives

$$-\Delta u_0 = E(u_0) u_0.$$

We define $\lambda_0 := E(u_0)$. This is the first (3).
(lowest) eigenvalue of $-\Delta$ on $W_0^{1,2}(U)$. Let now

$$\lambda_1 := \inf \left\{ \int_U |\nabla u|^2 dx \, ; \, u \in X, \; u \perp u_0 \right\}.$$

where $u \perp u_0$ means that $\int u u_0 \, dx = 0$. (*) see page (10) for a remark

Arguing as for the original minimization problem one can show that there is a minimizer u_1. To obtain that $-\Delta u_1 = \lambda_1 u_1$ we have to argue slightly differently:

If $v \in X$ with $v \perp u_0$. Arguing as previously we obtain that

$$\int_U \nabla u_1 \cdot \nabla v dx = \lambda_1 \int_U u_1 v \, dx$$

but from the previous step we know that $\int_U \nabla u_0 \cdot \nabla u_1 = \lambda_0 \int_U u_0 \cdot u_1 = 0$. It follows that $\int_U \nabla u_1 \cdot \nabla v dx = \lambda_1 \int u_1 \cdot v dx$ for all $v \in X$ and therefore arguing as previously it follows that $-\Delta u_1 = \lambda_1 u_1$.

Repeating this argument we can construct a sequence $(u_m)_{m \in \mathbb{N} \cup \{0\}}$ of eigenfuctions of $-\Delta$ with eigenvalues $(\lambda_m)_{m \in \mathbb{N} \cup \{0\}}$ where $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$

Claim (i) $\lambda_m \to \infty$ as $m \to \infty$.

(ii) $(u_m)_{m\in\mathbb{N}\cup\{0\}}$ is an orthonormal basis of $L^2(U)$.

Proof: (i) Assume that λ_m is bounded. Then since $\int_U |\nabla u_m|^2 dx = \lambda_m \int |u_m|^2 dx = \lambda_m$.

we obtain that u_m is a bounded sequence in $W_0^{1,2}(U)$. By the Rellich-Kondrachov compactness theorem u_m has a convergent subsequence in $L^2(U)$. But this contradicts the fact u_m is an orthonormal sequence in $L^2(U)$.

(ii) We will show that $\overline{\operatorname{span}\{(u_m)_{m\in\mathbb{N}\cup\{0\}}\}} \supset W_0^{1,2}(U)$. From this it will follow that

$\overline{\operatorname{span}\{(u_m)_{m\in\mathbb{N}\cup\{0\}}\}} = L^2(U)$ because $\overline{W_0^{1,2}(U)} \supset \overline{C_c(U)} = L^2(U)$ (for the proof of the last equality we refer to Adams Thm 2.19). Indeed

Assume that $\exists\, v \in W_0^{1,2}, v \notin \overline{\operatorname{span}\{(u_m)_{m\in\mathbb{N}\cup\{0\}}\}}$ with $\|v\| = 1$. Then $v \perp u_m$ $\forall m \in \mathbb{N}$. It follows that

$$\int_U |\nabla v|^2 dx \geq \int_U |\nabla u_m|^2 dx = \lambda_m, \ \forall m \in \mathbb{N} \text{ since } v \perp u_j, \forall j < m.$$

Since $v \in W_0^{1,2}(U)$ and $\lambda_m \to \infty$ this leads to a contradiction. (5)

This verifies the ansatz of separation of variables. Consider for example the heat equation

$$\begin{cases} v_t = \Delta v, \text{ on } U. \\ v|_{\partial U} = 0 \\ v(0,x) = f(x), \ f \in L^2(U). \end{cases}$$

(here $v : [0,\infty) \times \bar{U} \to \mathbb{R}$).

If $(u_m)_{m \in \mathbb{N}}$ is an orthonormal basis of $L^2(U)$ consisting of eigenfuctions of $-\Delta$, on $W_0^{1,2}(U)$, then $f = \sum_{m=1}^{\infty} a_m u_m$ where $(a_m)_{m \in \mathbb{N}} \subset \mathbb{R}$ with $\sum_{m=1}^{\infty} |a_m|^2 < \infty$.

Since $\begin{cases} w_t = \Delta w \\ v|_{\partial U} = 0 \\ w(0,x) = u_m \end{cases}$ has the solution

$w(t,x) = e^{-\lambda_m t} u_m(x)$, it follows, at least formally, that the original equation has solution

$v(t,x) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m t} u_m(x)$ (We have written at least formally as one has to check whether one can exchange series with differentiation)

Similarly the problem $\begin{cases} v_{tt} = \Delta v \\ v|_{\partial V} = 0 \\ v(0,x) = f(x) \end{cases}$.

has solution $v(t,x) = \sum_{m=1}^{\infty} a_m \cos(\sqrt{\lambda_m}\, t)\, u_m(x)$.

Therefore $\sqrt{\lambda_0}$ can be interpreted as the fundamental frequency of the drum.

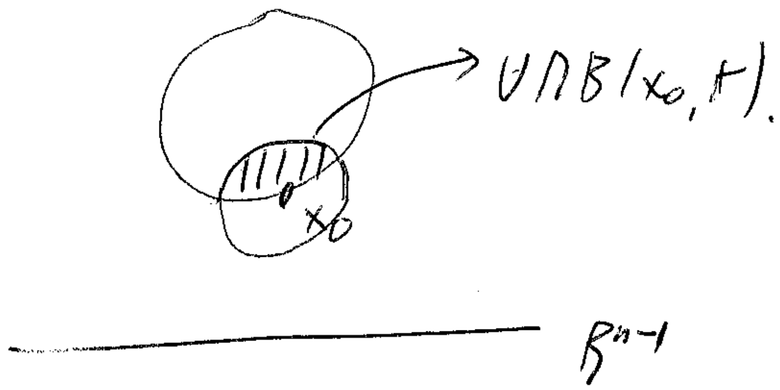
If $U \subseteq V$ then $\lambda_0(U) \geq \lambda_0(V)$. Indeed,

$$\lambda_0(U) = \inf_{\substack{u \in W_0^{1,p}(U) \\ \|u\|_{L^2} = 1}} \int_U |\nabla u|^2 dx \geq \inf_{\substack{u \in W_0^{1,p}(V) \\ \|u\|_{L^2} = 1}} \int_V |\nabla u|^2 dx = \lambda_0(V),$$

where the inequality follows from the fact that every function in $W_0^{1,p}(U)$ can be trivially extended to a function in $W_0^{1,p}(V)$ (as in the proof of the Rellich – Kondrachov compactness theorem).

Definition (C^k boundary). Let $k \in \mathbb{N}$, $U \subset \mathbb{R}^n$ open, bounded. We say that ∂U is C^k at $x_0 \in \partial U$ if there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon relabeling and reorienting the coordinate axes if necessary - we have

$$U \cap \underbrace{B}_{\text{Ball}}(\underbrace{x_0}_{\text{center}}, \underbrace{r}_{\text{radius}}) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$



We say that ∂U is C^k if it is C^k at every $x_0 \in \partial U$.

Remark. If ∂U is C^1, then we can define the outward pointing normal vector n_x at any point $x \in \partial U$. In particular if $u \in C^1(\bar{U})$ we can define the normal derivative $\dfrac{\partial u}{\partial n_x}$.

<u>Theorem</u> (Special case of Rellich-Kondrachov compactness, II.) Assume that $U \subseteq \mathbb{R}^n$ is open and bounded, and let $1 \leq p \leq \infty$. If ∂U is C^1, then the ~~identity map~~ imbedding $i: W^{1,p}(U) \to L^p(U)$ is compact.

<u>Remark:</u> The regularity of the boundary is needed to extend a $W^{1,p}(U)$ to a function $W_0^{1,p}(U+B(0,2))$ appropriately. After doing this the proof works as in the case of $W_0^{1,p}(U)$.

<u>Fourier transformation</u> Let $f \in L^1(\mathbb{R}^n)$ (real or complex valued). We define the Fourier trasformation of f by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi} f(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

<u>Theorem:</u> If $f \in C_c^\infty(\mathbb{R}^n)$ and α is a multindex then $\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi)$, $\xi \in \mathbb{R}^n$, where for $\xi = (\xi_1, \ldots, \xi_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. (Proof: Integration by parts).

Thm: If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ than $\hat{f} \in L^2(\mathbb{R}^n)$
and $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. (1)

Using this theorem we can define $\hat{f}$ for $f \in L^2(\mathbb{R}^n)$ in the sense of $L^2(\mathbb{R}^n)$ functions as follows: Since $f \in L^2(\mathbb{R}^n)$ there exists a sequence $f_m \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $f_m \to f$ in $L^2(\mathbb{R}^n)$
(for example $f_m = f \chi_{\{|f| \le m\}}$, where
$$\chi_{\{|f|\le m\}}(x) = \begin{cases} 1, & \text{if } |f(x)| \le m \\ 0, & \text{otherwise} \end{cases}).$$

Then f_m is Cauchy in $L^2(\mathbb{R}^n)$ and therefore by (1) so is $\hat{f}_m$. So there exists $g \in L^2$ with $\hat{f}_m \to g$. This g does not depend on the choice of the sequence and we define $\hat{f} = g$.

Thm (Plancherel). If $f \in L^2(\mathbb{R}^n)$ then
$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.

Proof: follows from (1) and the construction of $\hat{f}$ for $f \in L^2(\mathbb{R}^n)$.

Inverse Fourier transform: For $g \in L^1(\mathbb{R}^n)$ we define the inverse Fourier transform of g by $\check{g}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\xi} g(x)\, dx$. In a similar manner we can extend the definition for $g \in L^2(\mathbb{R}^n)$ and we have $\|\check{g}\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}$ for all $g \in L^2(\mathbb{R}^n)$.

(*) Remark on the proof of existence of an orthonormal basis consisting of eigenfunction of the Laplace. We note that for $u \in W_0^{1,2}(U)$ we have $u \perp u_0$ in $L^2(U) \Leftrightarrow u \perp u_0 \in W_0^{1,2}(U)$ because

$$\int_U \nabla u \cdot \nabla u_0 = \lambda_0 \int_U u \cdot u_0. \quad \text{so} \quad \int_U u \cdot u_0 = 0$$

if and only if $\int_U \nabla u \cdot \nabla u_0 + \int_U u \cdot u_0 = 0$.

Therefore $\{u \in X, u \perp u_0\}$ is closed in $W_0^{1,2}(U)$ and the previous arguments can be repeated.