1 Week 1

Brief summary of Lecture 1, April 18th The first lecture is going to be introductory and general. We will explain some boundary value problems and some eigenvalue problems arising in applications. We will explain some reasons that they are important. We will motivate the use of Sobolev spaces without really defining them: we will explain in what sense the spaces of differentiable functions are mostly not appropriate for the study of boundary and eigenvalue problems, as well as for differential equations in general.

Brief summary of Lecture 2, April 19th

Some notation

(i) In the entire lecture we denote by U an open subset of $\mathbb{R}^n$. $A \subset\subset U$ means that there exists a compact set K with $A \subset K \subset U$.

(ii) $C_c^\infty(U) := \{\phi : U \to \mathbb{R} \mid \phi \in C^\infty(U), \text{ and } \operatorname{supp}\phi \subset\subset U\}$, where $\operatorname{supp}\phi := \overline{\{x \in U : \phi(x) \neq 0\}}$. An element of $C_c^\infty(U)$ is called a <u>test function</u>.

(iii) A <u>multiindex</u> is a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$. For a multiindex α we define the <u>order</u> of α by $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We further define $\alpha! = \prod_{j=1}^n \alpha_j!$.

(iv) For $\phi \in C_c^\infty(U)$ and a multiindex α we define $D^\alpha\phi := \frac{\partial^{|\alpha|}\phi}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}}$.

(v) $L^p_{\text{loc}}(U) := \{\phi : U \to \mathbb{R} \mid \phi \in L^p(K) \text{ for all } K \subset\subset U\}$.

Definition 1.1 (**Weak derivatives**). *Suppose that $u, v \in L^1_{loc}(U)$. We say that v is the α^{th}-weak partial derivative of u, and write $D^\alpha u = v$ if*

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx, \forall \phi \in C_c^\infty(U).$$

Proposition 1.2 (Uniqueness of weak derivatives). *If $u \in L^1_{loc}(U)$ has a weak derivative then it is unique.*

Definition 1.3 (**Sobolev spaces**). *Let $k \in \mathbb{N}$ and $1 \le p \le \infty$. Then*

$$W^{k,p}(U) := \{u \in L^p(U) \mid D^\alpha u \text{ exists and } D^\alpha u \in L^p, \text{ for all multiindices } \alpha, \text{ with } |\alpha| \le k\}.$$

We equip $W^{k,p}(U)$ with the norm $\|.\|_{W^{k,p}(U)}$ defined by

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}}, & \text{if } p < \infty \\ \max_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty}, & \text{if } p = \infty. \end{cases}$$

Elementary properties of Sobolev spaces

Theorem 1.4 (Completeness). *For any $k \in \mathbb{N}$ and $1 \le p \le \infty$ the space $(W^{k,p}(U), \|.\|_{W^{k,p}(U)})$ is a Banach space.*

Theorem 1.5 (Leibnitz rule). *If $k \in \mathbb{N}$, $1 \le p \le \infty$, $u \in W^{k,p}(U)$ and $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and for all $|\alpha| \le k$ we have*

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u,$$

where $\beta \le \alpha$ means $\beta_j \le \alpha_j$ for all $j \in \{1, \dots, n\}$, and $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Theorem 1.6 (Approximation by smooth functions)**.** *Let $k \in \mathbb{N}$ and $p \in [1,\infty)$.*
(i) If $u \in W^{k,p}(\mathbb{R}^n)$, then there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{k,p}(\mathbb{R}^n)$.
(ii) If U is bounded and $u \in W^{k,p}(U)$, then there exists a sequence of functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.

2 Week 2

Brief summary of Lecture 3, April 25th The lecture will start by continuing the proof of approximation by smooth functions part I. To this end we will need the folloging lemma, which we are also going to prove:

Lemma 2.1. *1) Let $u \in L^1_{loc}(\mathbb{R}^n)$. Then $\eta_\epsilon * u \in C^\infty$. Here recall that $\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$ with $\eta(x) := \begin{cases} ke^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1. \end{cases}$, where $k > 0$ is chosen such that $\int_{R^n} \eta(x)dx = 1$.*

2) If $u \in L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$, then $\eta_\epsilon * u \in L^p(\mathbb{R}^n)$. Moreover $\|\eta_\epsilon * u\|_{L^p} \leq \|u\|_{L^p}$ and $\lim_{\epsilon\to 0+} \|\eta_\epsilon * u - u\|_{L^p} = 0$.

Definition 2.2. *Let X be a real (resp. complex) Banach space, $x \in X$ and x_n be a sequence in x. We say that x_n converges weakly to x and we write $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ for all continuous linear mappings $f : X \to \mathbb{R}$ (resp. $f : X \to \mathbb{C}$).*

Remark 1 (Corollary of Riesz representation theorem)**.** *If $(X, \langle .,. \rangle)$ is a Hilbert space then $x_n \rightharpoonup x$ if and only if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in X$.*

Theorem 2.3. *Let X be a Banach space. Then every weakly convergent sequence in X is bounded. Morover, if $x_n \rightharpoonup x$ then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

Theorem 2.4 (Banach-Alaoglu)**.** *Let X be a Hilbert space. Then every bounded sequence in X has a weakly convergent subsequence. The same conclusion holds if X is a reflexiv Banach space.*

Theorem 2.5. *Let X be a Hilbert space. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.*

Brief summary of Lecture 4, April 26th

Definition 2.6 (Compact operators)**.** *Let X, Y be Banach spaces and $K : X \to Y$ a linear operator. K is called compact, if for every bounded sequence x_n in X Kx_n has a convergent subsequence in Y.*

Theorem 2.7. *Let X, Y be Banach spaces, with X reflexiv, and $K : X \to Y$ be a linear operator. Then K is compact if and only if for any sequence x_n in X and any $x \in X$ we have $x_n \rightharpoonup x \implies Kx_n \to Kx$.*

Definition 2.8 ($W_0^{k,p}(U)$)**.** *Let $U \subset \mathbb{R}^n$ open. We denote by $W_0^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.*

Theorem 2.9 (Special case of Rellich-Kondrachov Compactness, I)**.** *Assume that $U \subset \mathbb{R}^n$ is bounded and open and let $1 \leq p \leq \infty$. Then the identity map $i : W_0^{1,p}(U) \to L^p(U)$ is compact.*

Definition 2.10 (C^k boundary)**.** *Let $k \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ open and bounded. We say that ∂U is C^k at $x_0 \in \partial U$ if there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^n \to \mathbb{R}$ such that -upon relabeling and reorienting the coordinate axes if necessary- we have*

$$U \cap B(x_0, r) = \{x \in B(x_0, r) \quad | \quad x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

Remark 2. *If ∂U is C^1, then we can define the outward pointing normal vector n_x at any point $x \in \partial U$. In particular for $u \in C^1(\overline{U})$ we can define the normal derivative $\frac{\partial u}{\partial n_x}$.*

Theorem 2.11 (Special case of Rellich-Kondrachov Compactness, II)**.** *Assume that $U \subset \mathbb{R}^n$ is bounded and open, and let $1 \leq p \leq \infty$. If ∂U is C^1, then the identity map $i : W^{1,p}(U) \to L^p(U)$ is compact.*

3 Week 3

Brief summary of Lecture 5, May 2nd

We will first finish the proof of the special case of the Rellich-Kondrachov compactness theorem, part I.

Let $U \subset \mathbb{R}^n$ open and bounded and let $X := \{u \in W_0^{1,2}(U) : \|u\|_{L^2(U)} = 1\}$. We consider the functional $E : X \to \mathbb{R}$, $E(u) = \int_U |\nabla u|^2 dx$. We are going to show that:

(i) The functional E has always a minimizer u_0.

(ii) $-\Delta u_0 = E(u_0)u_0$. In other words the minimizer is an eigenfunction of the Laplacian and the eigenvalue is the minimum of the functional E.

(iii) The eigenvectors of the Laplacian in $W_0^{1,2}(U)$ form an orthonormal basis of $L^2(U)$. We will explain why this verifies the validity of the ansatz of separation of variables for the equation of a drum and the heat equation on U with Dirichlet boundary conditions.

(iv) If $U \subset V$ then the minimum of E on U is bigger or equal than the minimum of E on V. As we will explain this can be interpreted in the way: "A big drum has a lower sound than a small drum".

In the rest of the lecture we will discuss the last week's material that we did not cover last week.

Brief summary of Lecture 6, May 3rd

The goal of this lecture is to get some first tools with the help of which one can show that a solution of an eigenvalue problem is smooth. In general this is considerably easier when there is no boundary and we will deal in this lecture with this case. If there is boundary things are considerably harder and we will deal with this issue in some weeks.

Definition 3.1 (Fourier Transformation)**.** *Let $f \in L^1(\mathbb{R}^n)$. We define the Fourier transformation of f by $\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix\xi} f(x)dx$, $\xi \in \mathbb{R}^n$.*

Theorem 3.2. *If $f \in C_c^\infty(\mathbb{R}^n)$ and α is a multiindex then $\widehat{D^\alpha f}(\xi) = (-i)^{|\alpha|}\xi^\alpha \hat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$. If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\hat{f} \in L^2(\mathbb{R}^n)$ and $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.*

Using the last theorem, one can define the Fourier transformation of an $L^2(\mathbb{R}^n)$ function uniquely in the space of $L^2(\mathbb{R}^n)$ functions.

Theorem 3.3 (Plancherel)**.** *If $f \in L^2(\mathbb{R}^n)$ then $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.*

4 Weeks 4-5

Brief summary of Lecture 7, May 9th

The biggest part of the lecture was on the summary of last week but since the plan has slightly changed we present it again.

Definition 4.1 (Inverse Fourier Transformation)**.** *Let $f \in L^1(\mathbb{R}^n)$. We define the Fourier transformation of f by $\check{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix\xi} f(x)dx$, $\xi \in \mathbb{R}^n$.*

In a similar manner as in the case of the Fourier transform we can define $\check{f}(\xi)$ for f in L^2. We then have that $\|\check{f}\|_{L^2} = \|f\|_{L^2}$.

Theorem 4.2 (Fourier inversion formula)**.** *If $f \in L^2(\mathbb{R}^n)$, then $f = \check{\hat{f}}$. In particular, if $\hat{f} \in L^1$ then $f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix\xi} f(\xi) d\xi$ for all $x \in \mathbb{R}^n$.*

Definition 4.3 (Fractional Sobolev spaces)**.** *Let $s \geq 0$. We define $H^s(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty\}$. We equip $H^s(\mathbb{R}^n)$ with the norm $\|f\|_{H^s(\mathbb{R}^n)} := (\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}}$.*

Theorem 4.4. *For every $s \geq 0$ the space $(H^s(\mathbb{R}^n), \|.\|_{H^s(\mathbb{R}^n)})$ is a Banach space.*

Remark 3. *The spaces $H^s(\mathbb{R}^n)$ can be analogously defined for $s < 0$ but in this case one needs to define carefully the Fourier transformation of f.*

Theorem 4.5. *(i) Let $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Then $f \in H^k(\mathbb{R}^n) \iff f \in W^{k,2}(\mathbb{R}^n)$. Moreover the norms $\|.\|_{H^k(\mathbb{R}^n)}$ and $\|.\|_{W^{k,2}(\mathbb{R}^n)}$ are equivalent.*
(ii) Let $s \geq 0$ and $f \in L^2(\mathbb{R}^n)$. Then $f \in H^{s+2}(\mathbb{R}^n)$ if and only if $\Delta f \in H^s(\mathbb{R}^n)$.

Theorem 4.6 (A Sobolev imbedding theorem)**.** *(i) If $s > \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n)$. Moreover there exists a constant $c > 0$ such that $\|f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in H^s(\mathbb{R}^n)$.*

(ii) Let $d \in \mathbb{N}$. If $s > \frac{n}{2} + d$ then $H^s(\mathbb{R}^n) \subset C^d(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that $\max_{|\alpha| \leq d} \|D^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in H^s(\mathbb{R}^n)$.

Brief summary of Lecture 8, May 9th

Application: regularity properties of eigenstates of Schrödinger operators in quantum mechanics.

(i) Let $v \in C_c^\infty(\mathbb{R}^3)$ be a spherically symmetric potential, and we assume that $-\Delta + v$ has an eigenfunction $\psi \in H^1(\mathbb{R}^3)$. Then $\psi \in C^\infty(\mathbb{R}^3)$.

(ii) We consider a function $\psi \in H^1(\mathbb{R}^3)$ with $-\Delta\psi - \frac{1}{|x|}\psi = E\psi$ for some real number E. We will show that $\psi \in C(\mathbb{R}^3)$. Moreover, $\psi \in C^\infty(\mathbb{R}^3/\{0\})$.

Now we return again to the Sobolev spaces $W^{1,p}(U)$, where in the rest of the lecture we assume that $U \subset \mathbb{R}^n$ is open and bounded.

Theorem 4.7 (Two Poincare inequalities)**.** *Assume that $1 \leq p \leq \infty$ and $U \subset \mathbb{R}^n$ is open and bounded. Then*
(i) Then there exists a constant C depending on U and p only such that $\|u\|_{L^p(U)} \leq C\|\nabla u\|_{L^p(U)}$, for all $u \in W_0^{1,p}(U)$.
(ii) If U is connected and ∂U is C^1, then there exists a constant C dending only on U and p such that $\|u - (u)_U\|_{L^p(U)} \leq C\|\nabla u\|_{L^p(U)}$, for all $u \in W^{1,p}(U)$. (Here $(u)_U$ denotes the average value of u in U.)

Brief summary of lecture 9, May 17th We will begin with the proof of the first part of Theorem 4.7 for $p = 2$.

Trace theorem The following theorem, gives us a way to define appropriately boundary conditions for functions in Sobolev spaves

Theorem 4.8 (Trace theorem)**.** *Assume that ∂U is C^1 and $1 \leq p < \infty$. Then there exists a unique linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ such that*
(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$, and
(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for each $u \in W^{1,p}(U)$, where the constant c depends on p and U.

Theorem 4.9. *Assume that $U \subset \mathbb{R}^n$ is open and bounded and that ∂U is C^1. Let $u \in W^{1,p}(U)$. Then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$.*

5 Brief summary of week 6, Introduction to second order elliptic partial differential equations

Note that we will comment first a little more on the proof of the trace theorem. In the next two weeks we will follow very closely the book of Evans Chapters 6.1, 6.2. We will begin the chapter of elliptic partial differential equations by motivating them with diffusion- reaction equations. This will be a big part of the first lecture. This will motivate also the definition of the operator L below:

Let $U \subset R^n$ be open and bounded. On the set of functions $u : U \to \mathbb{R}$, typically belonging to some Sobolev space, we consider a differential operator L having one of the following forms

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum b^i(x)u_{x_i} + c(x)u \tag{1}$$

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum b^i(x)u_{x_i} + c(x)u. \tag{2}$$

for some coefficient functions a^{ij}, b^i, c. The first form is called divergence form and the second non-divergence form. As we will explain in the motavation part, our aim is to study the boundary value problem

$$\begin{cases} Lu = f \text{ on } U \\ u = 0 \text{ on } \partial U. \end{cases} \tag{3}$$

We assume from now on, that

$$a^{ij}, b^i, c \in L^\infty(U),\ (i, j = 1, ..., n) \text{ and } f \in L^2. \tag{4}$$

.

Definition 5.1. *We say that the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{5}$$

for almost all $x \in U$ and all $\xi \in \mathbb{R}^n$.

Definition 5.2. *The bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ associated with the divergence form elliptic operator L defined by* (1) *is*

$$B[u, v] := \int_U \left(\sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx. \tag{6}$$

Definition 5.3. *We say that $u \in H_0^1(U)$ is a week solution of the boundary value problem* (3) *if $B[u, v] = \int fvdx$ for all $v \in H_0^1(U)$.*

With the next few theorems, we aim to develope some tools fpr showing existence and uniqueness of week solutions of boundary value problems. We assume that $(H, \|.\|)$ is a real HIlbert space with inner product $(.,.)$.

Theorem 5.4 (Lax-Millgram Theorem)**.** *Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping for which there exist constants $\alpha, \beta > 0$ such that $|B[u, v]| \leq \alpha\|u\|\|v\|$ for all $u, v \in H$ and $\beta\|u\|^2 \leq B[u, u]$ for all $u \in H$. Let $f : H \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique element $u \in H$ such that $B[u, v] = f(v)$ for all $v \in H$.*

6 Week 7, Theorems on existence of weak solutions of elliptic boundary value problems, Fredholm alternative

Theorem 6.1. *Let B be the bilinear form defined by* (6) *(assuming* (4)*). Then there exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that $|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$ and $\beta \|u\|^2_{H_0^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$, for all $u, v \in H_0^1(U)$.*

Theorem 6.2 (First existence Theorem for weak solutions)**.** *There exists $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each $f \in L^2(U)$ there exsists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem*

$$\begin{cases} Lu + \mu u = f \ \text{ on } U \\ u = 0 \ \text{ on } \partial U. \end{cases} \tag{7}$$

In the rest of the lecture $(H, (.,.))$ be a Hilbert space.

If $K : H \to H$ a bounded operator. Then for any $u \in H$ there exists $w \in H$ such that $(u, Kv) = (w, v)$, for all $v \in H$.

Definition 6.3. *Let $K : H \to H$ be a linear bounded operator. We define $K^* : H \to H$ by $K^*u = w$, where u, w are as above.*

Lemma 6.4. *Let $K : H \to H$ be a linear compact operator. Then K^* is also a linear compact operator.*

For an operator A we denote by $N(A)$ the nullspace (or Kernel) of A and by $R(A)$ the range of A.

Theorem 6.5 (Fredholm alternative)**.** *Let $K : H \to H$ be a linear compact operator. Then*
(i) $N(I - K)$ is finite dimensional.
(ii) $R(I - K)$ is closed.
(iii) $R(I - K) = N(I - K^*)$.
(iv) $N(I - K) = \{0\}$ iff $R(I - K) = \{H\}$.
(v) $\dim N(I - K) = \dim N(I - K^*)$.

Definition 6.6. *For an elliptic differential operator L given by* (1) *we define L^*, the formal adjoint of L, given by*

$$L^*u = -\sum_{i,j=1}^{n} (a^{ij}(x) u_{x_j})_{x_i} - \sum b^i(x) u_{x_i} + \left(c(x) - \sum_{i=1}^{n} b_{i,x_i} \right) u, \tag{8}$$

provided that the functions b_i are in $C^1(U)$.

Theorem 6.7 (Second existence Theorem for weak solutions)**.** *(i) Precisely one of the following statements holds:*
(a) for each $f \in L^2(U)$ there exists a unique weak solution u of the boundary-value problem

$$\begin{cases} Lu = f \ \text{ in } U \\ u = 0 \ \text{ on } \partial U. \end{cases} \tag{9}$$

(b) There exists a weak solution $u \neq 0$ of the homogeneous problem

$$\begin{cases} Lu = 0 \ \text{ in } U \\ u = 0 \ \text{ on } \partial U. \end{cases} \tag{10}$$

(ii) If assertion (b) holds, then the dimesion of the subspace $N \subset H_0^1(U)$ of weak solutions of (10) is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of

$$\begin{cases} L^* u = 0 \text{ in } U \\ u = 0 \text{ on } \partial U. \end{cases} \tag{11}$$

(iii) The boundary value problem (9) has a solution if and only if $(f, v) = 0$, for all $v \in N^*$.

Theorem 6.8 (Third existence Theorem for weak solutions)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, and L be uniformly elliptic. Then there exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary value problem*

$$\begin{cases} Lu = \lambda u + f \text{ in } U \\ u = 0 \text{ on } \partial U. \end{cases} \tag{12}$$

has a unique solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$. If Σ is infinite, then Σ consists of an increasing sequence λ_k with $\lambda_k \to \infty$.

If $U \subset \mathbb{R}^n$ is open and bounded then we define $H^m(U); = W^{2,m}(U), H^m_{loc}(U); = W^{2,m}_{loc}(U)$.

Theorem 6.9 (Interior regularity)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Let m be a nonnegative integer and L be an elliptic operator given by (1). Assume that $a^{ij}, b^j, c \in C^{m+1}(U), i, j = 1, \dots, n$ and $f \in H^m(U)$. Suppose moreover, that $u \in H^1(U)$ and in the sense of weak derivatives we have $Lu = f$ in U. Then $u \in H^{m+2}_{loc}(U)$ and if $V \subset\subset U$ then*

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^{m+2}(U)} + \|u\|_{L^2(U)}), \tag{13}$$

where $C > 0$ depends only on L,m,U and V.

Corollary 6.10 (Infinite differentiability in the interior)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Let m be a nonnegative integer and L be an elliptic operator given by (1). Assume that $a^{ij}, b^j, c \in C^\infty(U), i, j = 1, \dots, n$ and $f \in C^\infty(U)$. Suppose moreover, that $u \in H^1(U)$ and in the sense of weak derivatives we have $Lu = f$ in U. Then $u \in C^\infty(U)$.*

Theorem 6.11 (Boundary regularity)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Let m be a nonnegative integer and L be an elliptic operator given by (1). Assume that $a^{ij}, b^j, c \in C^{m+1}(\overline{U}), i, j = 1, \dots, n$ and $f \in H^m(U)$. Suppose moreover, that $u \in H_0^1(U)$ is a weak solution of (9). Assume finally that*

$$\partial U \in C^{m+2}.$$

Then $u \in H^{m+2}(U)$ and

$$\|u\|_{H^{m+2}(U)} \leq C(\|f\|_{H^{m+2}(U)} + \|u\|_{L^2(U)}), \tag{14}$$

where C depends only on U, L and m.

Theorem 6.12 (Infinite differentiability up to the boundary)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Let m be a nonnegative integer and L be an elliptic operator given by (1). Assume that $a^{ij}, b^j, c \in C^\infty(\overline{U}), i, j = 1, \dots, n$ and $f \in C^\infty(\overline{U})$. Suppose moreover, that $u \in H_0^1(U)$ is a weak solution of (9). Assume finally that*

$$\partial U \in C^\infty.$$

Then $u \in C^\infty(\overline{U})$.

7 Maximum Principles

This section discusses conditions under which solutions of elliptic boundary value problems attain their minimum or maximum at the boundary. Below L is the operator given by (2). We will always assume that a^{ij}, b^i, c are continuous. Without loss of generality we also assume the symmetry condition $a^{ij} = a^{ji}$, $i, j = 1, \ldots, n$. We also assume that U is open and bounded.

We will first prove the following preliminary lemma:

Lemma 7.1. *If two matrices $A, B \in R^{n\times n}$ are symmetric and positive definite then $Tr(AB) \geq 0$.*

Theorem 7.2 (Weak maximum principle)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U.*
(i) If $Lu \leq 0$ in U then $\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x)$.
(ii) If $Lu \geq 0$ in U, then $\min_{x\in\overline{U}} u(x) = \min_{x\in\partial U} u(x)$.

The following theorem is a generalization of the previous when $c \geq 0$. We define $u^+ = max(u, 0)$ and $u^- = -min(u, 0)$.

Theorem 7.3 (Weak maximum principle for $c \geq 0$)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U.*
(i) If $Lu \leq 0$ in U then $\max_{x\in\overline{U}} u(x) \leq \max_{x\in\partial U} u^+(x)$.
(ii) If $Lu \geq 0$ in U, then $\min_{x\in\overline{U}} u(x) \geq -\max_{x\in\partial U} u^-(x)$.
In particular if $Lu = 0$ in U, then $\max_{x\in\overline{U}} |u(x)| = \max_{x\in\partial U} |u(x)|$.

The following lemma has the goal to strengthen the weak maximum principles.

Definition 7.4. *Let $x^0 \in \partial U$. We say that U satisfies the interior ball condition at x^0 if there is an open ball $B \subset U$ such that $x^0 \in \partial B$.*

Lemma 7.5 (Hopf's Lemma)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$. Suppose that $Lu \leq 0$ in U, and that there exists a point $x^0 \in \partial U$ such that*

$$u(x^0) > u(x), \quad \forall x \in U.$$

Assume that U satisfies the interior ball condition at x^0. Then
(i) If $c = 0$ in U then $\frac{\partial u}{\partial n}(x^0) > 0$, where n is the outer unit normal to B.
(ii) If $c \geq 0$ in U and $u(x^0) \geq 0$ then the same conclusion holds.

Theorem 7.6 (Strong maximum principle)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U. Suppose also that U is connected open and bounded.*

(i) If $Lu \leq 0$ in U and u attains its maximum over $\overline{U}$ at an interior point, then u is constant within U.

(ii) If $Lu \geq 0$ in U and u attains its minimum over $\overline{U}$ at an interior point, then u is constant within U.

Theorem 7.7 (Strong maximum principle for $c \geq 0$)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U. Suppose also that U is connected open and bounded.*

(i) If $Lu \leq 0$ in U and u attains a nonnegative maximum over $\overline{U}$ at an interior point, then u is constant within U.

(ii) If $Lu \geq 0$ in U and u attains a nonpositive minimum over $\overline{U}$ at an interior point, then u is constant within U.

8 Eigenvalues and Eigenfunctions of symmetric elliptic operators

In this chapter we will always consider an elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j}, \tag{15}$$

where $a^{ij} \in C^\infty(\overline{U})$. We will also always assume that $a^{ij} = a^{ji}$ and that U is open and bounded.

We will investigate properties of the eigenvalues and eigenfunctions of L in $H_0^1(U)$. We will prove uniqueness and positivity of the eigenfunction to the smallest eigenvalue.

We will start with a theorem which we are going to need below

Theorem 8.1 (Spectral theorem for compact self-adjoint operators)**.** *Let H be a separable Hilbert space, and suppose $K : H \to H$ is compact and self-adjoint. Then there exists a (countable) orthonomal basis of H consisting of eigenvectors of K. Moreover, the eigenvalues of K converge to zero.*

Theorem 8.2 (Eigenvalues of symmetric elliptic operators)**.** *Let U, L be as above and we consider the eigenvalue problem $Lu = \lambda u$ in $H_0^1(U)$.*

(i) Each eigenvalue of L is positive.

(ii) Furthermore, if we repeat each eigenvalue according to its (finite) multiplicity, we have $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$, where

$$0 < \lambda_1 \le \lambda_2 \le \dots$$

and $\lambda_k \to \infty$ as $k \to \infty$.

(iii) There exists an orthonormal basis $\{w_k\}$ of $L^2(U)$, where $w_k \in H_0^1(U)$ is an eigenfunction of L corresponding to λ_k.

Definition 8.3. *We call $\lambda_1 > 0$ the principal eigenvalue of L.*

Theorem 8.4 (Variational principle for the principal eigenvalue)**.** *(i) If B is the bilinear form associated to L then*

$$\lambda_1 = \min\{B[u,u] : u \in H_0^1(U), \|u\|_{L^2} = 1\}.$$

(ii) The above minimum is attained for a function w_1, positive within U, which is eigenfunction of L with eigenvalue λ_1.

(iii)(Uniqueness of eigenfunction to the principal eigenvalue) Any other eigenfunction of L to the eigenvalue λ_1 is a multiple of w_1.