Brief summary of Lecture 1, April 18th The first lecture is going to be introductory and general. We will explain some boundary value problems and some eigenvalue problems arising in applications. We will explain some reasons that they are important. We will motivate the use of Sobolev spaces without really defining them: we will explain in what sense the spaces of differentiable functions are mostly not appropriate for the study of boundary and eigenvalue problems, as well as for differential equations in general.

Brief summary of Lecture 2, April 19th

Some notation

(i) In the entire lecture we denote by U an open subset of $\mathbb{R}^n$. $A \subset\subset U$ means that there exists a compact set K with $A \subset K \subset U$.

(ii) $C_c^\infty(U) := \{\phi : U \to \mathbb{R} \mid \phi \in C^\infty(U), \text{ and } \operatorname{supp}\phi \subset\subset U\}$, where $\operatorname{supp}\phi := \overline{\{x \in U : \phi(x) \neq 0\}}$. An element of $C_c^\infty(U)$ is called a test function.

(iii) A multiindex is a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$. For a multiindex α we define the order of α by $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We further define $\alpha! = \prod_{j=1}^n \alpha_j!$.

(iv) For $\phi \in C_c^\infty(U)$ and a multiindex α we define $D^\alpha \phi := \frac{\partial^{|\alpha|}\phi}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$.

(v) $L^p_{\text{loc}}(U) := \{\phi : U \to \mathbb{R} \mid \phi \in L^p(K) \text{ for all } K \subset\subset U\}$.

Definition 1 (**Weak derivatives**). *Suppose that $u, v \in L^1_{loc}(U)$. We say that v is the α^{th}-weak partial derivative of u, and write $D^\alpha u = v$ if*

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx, \forall \phi \in C_c^\infty(U).$$

Proposition 0.1 (Uniqueness of weak derivatives). *If $u \in L^1_{loc}(U)$ has a weak derivative then it is unique.*

Definition 2 (**Sobolev spaces**). *Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then*

$$W^{k,p}(U) := \{u \in L^p(U) \mid D^\alpha u \text{ exists and } D^\alpha u \in L^p, \text{ for all multiindices } \alpha, \text{ with } |\alpha| \leq k\}.$$

We equip $W^{k,p}(U)$ with the norm $\|.\|_{W^{k,p}(U)}$ defined by

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}}, & \text{if } p < \infty \\ \max_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty}, & \text{if } p = \infty. \end{cases}$$

Elementary properties of Sobolev spaces

Theorem 0.2 (Completeness). *For any $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ the space $(W^{k,p}(U), \|.\|_{W^{k,p}(U)})$ is a Banach space.*

Theorem 0.3 (Leibnitz rule). *If $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $u \in W^{k,p}(U)$ and $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and for all $|\alpha| \leq k$ we have*

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u,$$

where $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ for all $j \in \{1, \ldots, n\}$, and $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Theorem 0.4 (Approximation by smooth functions). *Let $k \in \mathbb{N}$ and $p \in [1, \infty)$.*

(i) If $u \in W^{k,p}(\mathbb{R}^n)$, then there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{k,p}(\mathbb{R}^n)$.

(ii) If U is bounded and $u \in W^{k,p}(U)$, then there exists a sequence of functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.