**Brief summary of Lecture 3, April 25th** The lecture will start by continuing the proof of approximation by smooth functions part I. To this end we will need the folloging lemma, which we are also going to prove:

**Lemma 0.1.** *1) Let $u \in L^1_{loc}(\mathbb{R}^n)$. Then $\eta_\epsilon * u \in C^\infty$. Here recall that $\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$ with* $\eta(x) := \begin{cases} ke^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1. \end{cases}$ *, where $k > 0$ is chosen such that $\int_{R^n} \eta(x)dx = 1$.*

*2) If $u \in L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$, then $\eta_\epsilon * u \in L^p(\mathbb{R}^n)$. Moreover $\|\eta_\epsilon * u\|_{L^p} \leq \|u\|_{L^p}$ and $\lim_{\epsilon\to 0+} \|\eta_\epsilon * u - u\|_{L^p} = 0$.*

**Definition 1.** *Let $X$ be a real (resp. complex) Banach space, $x \in X$ and $x_n$ be a sequence in $x$. We say that $x_n$ converges weakly to $x$ and we write $x_n \rightharpoonup x$ , if $f(x_n) \to f(x)$ for all continuous linear mappings $f : X \to \mathbb{R}$ (resp. $f : X \to \mathbb{C}$).*

**Remark 1** (Corollary of Riesz representation theorem)**.** *If $(X, \langle ., . \rangle)$ is a Hilbert space then $x_n \rightharpoonup x$ if and only if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in X$.*

**Theorem 0.2.** *Let $X$ be a Banach space. Then every weakly convergent sequence in $X$ is bounded. Morover, if $x_n \rightharpoonup x$ then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

**Theorem 0.3** (Banach-Alaoglu)**.** *Let $X$ be a Hilbert space. Then every bounded sequence in $X$ has a weakly convergent subsequence. The same conclusion holds if $X$ is a reflexiv Banach space.*

**Theorem 0.4.** *Let $X$ be a Hilbert space. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.*

**Brief summary of Lecture 4, April 26th**

**Definition 2** (Compact operators)**.** *Let $X, Y$ be Banach spaces and $K : X \to Y$ a linear operator. $K$ is called compact, if for every bounded sequence $x_n$ in $X$ $Kx_n$ has a convergent subsequence in $Y$.*

**Theorem 0.5.** *Let $X, Y$ be Banach spaces, with $X$ reflexiv, and $K : X \to Y$ be a linear operator. Then $K$ is compact if and only if for any sequence $x_n$ in $X$ and any $x \in X$ we have $x_n \rightharpoonup x \implies Kx_n \to Kx$.*

**Definition 3** ($W^{k,p}_0(U)$)**.** *Let $U \subset \mathbb{R}^n$ open. We denote by $W^{k,p}_0(U)$ the closure of $C^\infty_c(U)$ in $W^{k,p}(U)$.*

**Theorem 0.6** (Special case of Rellich-Kondrachov Compactness, I)**.** *Assume that $U \subset \mathbb{R}^n$ is bounded and open and let $1 \leq p \leq \infty$. Then the identity map $i : W^{1,p}_0(U) \to L^p(U)$ is compact.*

**Definition 4** ($C^k$ boundary)**.** *Let $k \in \mathbb{N}$ and $U \subset \mathbb{R}^n$ open and bounded. We say that $\partial U$ is $C^k$ at $x_0 \in \partial U$ if there exists $r > 0$ and a $C^k$ function $\gamma : \mathbb{R}^n \to \mathbb{R}$ such that -upon relabeling and reorienting the coordinate axes if necessary- we have*

$$U \cap B(x_0, r) = \{x \in B(x_0, r) \quad | \quad x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

**Remark 2.** *If $\partial U$ is $C^1$, then we can define the outward pointing normal vector $n_x$ at any point $x \in \partial U$. In particular for $u \in C^1(\overline{U})$ we can define the normal derivative $\frac{\partial u}{\partial n_x}$.*

**Theorem 0.7** ( Special case of Rellich-Kondrachov Compactness, II)**.** *Assume that $U \subset \mathbb{R}^n$ is bounded and open, and let $1 \leq p \leq \infty$. If $\partial U$ is $C^1$, then the identity map $i : W^{1,p}(U) \to L^p(U)$ is compact.*