Brief summary of Lecture 5, May 2nd

We will first finish the proof of the special case of the Rellich-Kondrachov compactness theorem, part I.

Let $U \subset \mathbb{R}^n$ open and bounded and let $X := \{u \in W_0^{1,2}(U) : \|u\|_{L^2(U)} = 1\}$. We consider the functional $E : X \to \mathbb{R}$, $E(u) = \int_U |\nabla u|^2 dx$. We are going to show that:

(i) The functional E has always a minimizer u_0.

(ii) $-\Delta u_0 = E(u_0)u_0$. In other words the minimizer is an eigenfunction of the Laplacian and the eigenvalue is the minimum of the functional E.

(iii) The eigenvectors of the Laplacian in $W_0^{1,2}(U)$ form an orthonormal basis of $L^2(U)$. We will explain why this verifies the validity of the ansatz of separation of variables for the equation of a drum and the heat equation on U with Dirichlet boundary conditions.

(iv) If $U \subset V$ then the minimum of E on U is bigger or equal than the minimum of E on V. As we will explain this can be interpreted in the way: "A big drum has a lower sound than a small drum".

In the rest of the lecture we will discuss the last week's material that we did not cover last week.

Brief summary of Lecture 6, May 3rd

The goal of this lecture is to get some first tools with the help of which one can show that a solution of an eigenvalue problem is smooth. In general this is considerably easier when there is no boundary and we will deal in this lecture with this case. If there is boundary things are considerably harder and we will deal with this issue in some weeks.

Definition 1 (Fourier Transformation). *Let $f \in L^1(\mathbb{R}^n)$. We define the Fourier transformation of f by $\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix\xi} f(x)dx$, $\xi \in \mathbb{R}^n$.*

Theorem 0.1. *If $f \in C_c^\infty(\mathbb{R}^n)$ and α is a multiindex then $\widehat{D^\alpha f}(\xi) = (-i)^{|\alpha|}\xi^\alpha \hat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$. If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\hat{f} \in L^2(\mathbb{R}^n)$ and $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.*

Using the last theorem, one can define the Fourier transformation of an $L^2(\mathbb{R}^n)$ function uniquely in the space of $L^2(\mathbb{R}^n)$ functions.

Theorem 0.2 (Plancherel). *If $f \in L^2(\mathbb{R}^n)$ then $\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$.*

Definition 2 (Fractional Sobolev spaces). *Let $s \geq 0$. We define $H^s(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty\}$. We equip $H^s(\mathbb{R}^n)$ with the norm $\|f\|_{H^s(\mathbb{R}^n)} := (\int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}}$.*

Theorem 0.3. *For every $s \geq 0$ the space $(H^s(\mathbb{R}^n), \|.\|_{H^s(\mathbb{R}^n)})$ is a Banach space.*

Remark 1. *The spaces $H^s(\mathbb{R}^n)$ can be analogously defined for $s < 0$ but in this case one needs to define carefully the Fourier transformation of f.*

Theorem 0.4. *(i) Let $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Then $f \in H^k(\mathbb{R}^n) \iff f \in W^{k,2}(\mathbb{R}^n)$. Moreover the norms $\|.\|_{H^k(\mathbb{R}^n)}$ and $\|.\|_{W^{k,2}(\mathbb{R}^n)}$ are equivalent.*
(ii) Let $s \geq 0$ and $f \in L^2(\mathbb{R}^n)$. Then $f \in H^{s+2}(\mathbb{R}^n)$ if and only if $\Delta f \in H^s(\mathbb{R}^n)$.

Theorem 0.5 (A Sobolev imbedding theorem). *(i) If $s > \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n)$. Moreover there exists a constant $c > 0$ such that $\|f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in H^s(\mathbb{R}^n)$.*

(ii) Let $d \in \mathbb{N}$. If $s > \frac{n}{2} + d$ then $H^s(\mathbb{R}^n) \subset C^d(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that $\max_{|\alpha| \leq d} \|D^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in H^s(\mathbb{R}^n)$.

Application: regularity properties of eigenstates of Schrödinger operators in quantum mechanics.

(i) We consider a function $\psi \in H^1(\mathbb{R}^3)$ with $-\Delta\psi - \frac{1}{|x|}\psi = E\psi$ for some real number E. We will show that $\psi \in C(\mathbb{R}^3)$.

(ii) Let $v \in C_c^\infty(\mathbb{R}^3)$ be a spherically symmetric potential, and we assume that $-\Delta + v$ has an eigenfunction $\psi \in H^1(\mathbb{R}^3)$. Then $\psi \in C^\infty(\mathbb{R}^3)$.