1 Brief summary of week 6, Introduction to second order elliptic partial differential equations

Note that we will comment first a little more on the proof of the trace theorem. In the next two weeks we will follow very closely the book of Evans Chapters 6.1, 6.2. We will begin the chapter of elliptic partial differential equations by motivating them with diffusion- reaction equations. This will be a big part of the first lecture. This will motivate also the definition of the operator L below:

Let $U \subset R^n$ be open and bounded. On the set of functions $u : U \to \mathbb{R}$, typically belonging to some Sobolev space, we consider a differential operator L having one of the following forms

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum b^i(x)u_{x_i} + c(x)u \tag{1}$$

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum b^i(x)u_{x_i} + c(x)u. \tag{2}$$

for some coefficient functions a^{ij}, b^i, c. The first form is called divergence form and the second non-divergence form. As we will explain in the motavation part, our aim is to study the boundary value problem

$$\begin{cases} Lu = f \text{ on } U \\ u = 0 \text{ on } \partial U. \end{cases} \tag{3}$$

We assume from now on, that

$$a^{ij}, b^i, c \in L^\infty(U), \ (i,j = 1, ..., n) \text{ and } f \in L^2. \tag{4}$$

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Definition 1.1. *We say that the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{5}$$

for almost all $x \in U$ and all $\xi \in \mathbb{R}^n$.

Definition 1.2. *The bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ associated with the divergence form elliptic operator L defined by* (1) *is*

$$B[u,v] := \int_U \left(\sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx. \tag{6}$$

Definition 1.3. *We say that $u \in H_0^1(U)$ is a week solution of the boundary value problem* (3) *if $B[u,v] = \int fvdx$ for all $v \in H_0^1(U)$.*

With the next few theorems, we aim to develope some tools fpr showing existence and uniqueness of week solutions of boundary value problems. We assume that $(H, \|.\|)$ is a real HIlbert space with inner product $(.,.)$.

Theorem 1.4 (Lax-Millgram Theorem)**.** *Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping for which there exist constants $\alpha, \beta > 0$ such that $|B[u,v]| \leq \alpha\|u\|\|v\|$ for all $u, v \in H$ and $\beta\|u\|^2 \leq B[u,u]$ for all $u \in H$. Let $f : H \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique element $u \in H$ such that $B[u,v] = f(v)$ for all $v \in H$.*

Theorem 1.5. *Let B be the bilinear form defined by* (6) *(assuming* (4)*). Then there exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that $|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$ and $\beta \|u\|^2_{H_0^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$, for all $u, v \in H_0^1(U)$.*

Theorem 1.6 (First existence Theorem for weak solutions)**.** *There exists $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each $f \in L^2(U)$ there exsists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem*

$$\begin{cases} Lu + \mu u = f \ \text{ on } U \\ u = 0 \ \text{ on } \partial U. \end{cases} \tag{7}$$