Brief summary of Lecture 7, May 9th

The biggest part of the lecture was on the summary of last week but since the plan has slightly changed we present it again.

Definition 1 (Inverse Fourier Transformation)**.** *Let $f \in L^1(\mathbb{R}^n)$. We define the Fourier transformation of f by $\check{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix\xi} f(x)dx$, $\xi \in \mathbb{R}^n$.*

In a similar manner as in the case of the Fourier transform we can define $\check{f}(\xi)$ for f in L^2. We then have that $\|\check{f}\|_{L^2} = \|f\|_{L^2}$.

Theorem 0.1 (Fourier inversion formula)**.** *If $f \in L^2(\mathbb{R}^n)$, then $f = \check{\hat{f}}$. In particular, if $\hat{f} \in L^1$ then $f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix\xi} f(\xi)d\xi$ for all $x \in \mathbb{R}^n$.*

Definition 2 (Fractional Sobolev spaces)**.** *Let $s \geq 0$. We define $H^s(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : \int(1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty\}$. We equip $H^s(\mathbb{R}^n)$ with the norm $\|f\|_{H^s(\mathbb{R}^n)} := (\int(1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}}$.*

Theorem 0.2. *For every $s \geq 0$ the space $(H^s(\mathbb{R}^n), \|.\|_{H^s(\mathbb{R}^n)})$ is a Banach space.*

Remark 1. *The spaces $H^s(\mathbb{R}^n)$ can be analogously defined for $s < 0$ but in this case one needs to define carefully the Fourier transformation of f.*

Theorem 0.3. *(i) Let $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Then $f \in H^k(\mathbb{R}^n) \iff f \in W^{k,2}(\mathbb{R}^n)$. Moreover the norms $\|.\|_{H^k(\mathbb{R}^n)}$ and $\|.\|_{W^{k,2}(\mathbb{R}^n)}$ are equivalent.*
(ii) Let $s \geq 0$ and $f \in L^2(\mathbb{R}^n)$. Then $f \in H^{s+2}(\mathbb{R}^n)$ if and only if $\Delta f \in H^s(\mathbb{R}^n)$.

Theorem 0.4 (A Sobolev imbedding theorem)**.** *(i) If $s > \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n)$. Moreover there exists a constant $c > 0$ such that $\|f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in H^s(\mathbb{R}^n)$.*

(ii) Let $d \in \mathbb{N}$. If $s > \frac{n}{2} + d$ then $H^s(\mathbb{R}^n) \subset C^d(\mathbb{R}^n)$. Moreover, there exists a constant $c > 0$ such that $\max_{|\alpha| \leq d} \|D^\alpha f\|_{L^\infty(\mathbb{R}^n)} \leq c\|f\|_{H^s(\mathbb{R}^n)}$ for all $f \in H^s(\mathbb{R}^n)$.

Brief summary of Lecture 8, May 9th

Application: regularity properties of eigenstates of Schrödinger operators in quantum mechanics.

(i) Let $v \in C_c^\infty(\mathbb{R}^3)$ be a spherically symmetric potential, and we assume that $-\Delta + v$ has an eigenfunction $\psi \in H^1(\mathbb{R}^3)$. Then $\psi \in C^\infty(\mathbb{R}^3)$.

(ii) We consider a function $\psi \in H^1(\mathbb{R}^3)$ with $-\Delta\psi - \frac{1}{|x|}\psi = E\psi$ for some real number E. We will show that $\psi \in C(\mathbb{R}^3)$. Moreover, $\psi \in C^\infty(\mathbb{R}^3/\{0\})$.

Now we return again to the Sobolev spaces $W^{1,p}(U)$, where in the rest of the lecture we assume that $U \subset \mathbb{R}^n$ is open and bounded.

Theorem 0.5 (Two Poincare inequalities)**.** *Assume that $1 \leq p \leq \infty$ and $U \subset \mathbb{R}^n$ is open and bounded. Then*
(i) Then there exists a constant C depending on U and p only such that $\|u\|_{L^p(U)} \leq C\|\nabla u\|_{L^p(U)}$, for all $u \in W_0^{1,p}(U)$.
(ii) If U is connected and ∂U is C^1, then there exists a constant C dending only on U and p such that $\|u - (u)_U\|_{L^p(U)} \leq C\|\nabla u\|_{L^p(U)}$, for all $u \in W^{1,p}(U)$. (Here $(u)_U$ denotes the average value of u in U.)

Brief summary of lecture 9, May 17th We will begin with the proof of the first part of Theorem 0.5 for $p = 2$.

Trace theorem The following theorem, gives us a way to define appropriately boundary conditions for functions in Sobolev spaves

Theorem 0.6 (Trace theorem)**.** *Assume that ∂U is C^1 and $1 \leq p < \infty$. Then there exists a unique linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ such that*
(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$, and
(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for each $u \in W^{1,p}(U)$, where the constant c depends on p and U.

Theorem 0.7. *Assume that $U \subset \mathbb{R}^n$ is open and bounded and that ∂U is C^1. Let $u \in W^{1,p}(U)$. Then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$.*