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Institute for Analysis
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Exam

Classical methods for partial differential equations.

Note: To pass the exam you need 16 out of 48 points.

Exercise 1 (4 + 3 + 5 points)

a) Let $U \subset \mathbb{R}^n$ be open and bounded. On the set of functions $u : \overline{U} \to \mathbb{R}$ with $u \in C(\overline{U}) \cap C^2(U)$, we consider an elliptic differential operator L having the form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i},$$

for some functions $a^{ij}, b^i \in C(U)$. Prove that if $Lu < 0$ in U, then the maximum of u in $\overline{U}$ is attained on ∂U.

b) Let $u = u(x, y)$ be a continuous function on the disc $\overline{B_R(0)} \subset \mathbb{R}^2$, which is also harmonic on $B_R(0)$. Assume that $u(R\cos(\theta), R\sin(\theta)) = 1 + 5\cos^2(\theta)$, for $\theta \in [0, 2\pi]$. Determine the minimum of u on $\overline{B_R(0)}$ and explain on how many points it is attained.

c) Let $\Omega \subset \mathbb{R}^n$ be a C^1 domain and assume that G is a Green's function in Ω. Prove that if $u \in C^2(\overline{\Omega})$, then for all $x \in \Omega$ we have

$$u(x) = \int_{\Omega} G(x, y)(-\Delta u(y))dy + \int_{\partial\Omega} -u(y)\nabla_y G(x, y) \cdot \nu(y) d\sigma_y.$$

Exercise 2 (6 + 6 points)

a) Let $f \in L^1(\mathbb{R}^n)$ and Φ be the fundamental solution of the heat equation in $\mathbb{R}^n$. Define $u(t, x) = \int_{\mathbb{R}^n} \Phi(t, x - y) f(y)dy$. Prove that $\lim_{t\to 0+} \|u(t, .) - f\|_{L^1(\mathbb{R}^n)} = 0$.

Hint: You may use without proof that $\lim_{h\to 0} \|\tau_h f - f\|_{L^1} = 0$, where $(\tau_h f)(x) := f(x + h)$.

b) Recall that if $f \in C^{0,1}([0, \infty) \times \mathbb{R})$ and $g \in C^1(\mathbb{R})$ then the solution of

$$\begin{cases} u_t + bu_x = f(t, x), & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

is explicitly given by $u(t, x) = g(x - bt) + \int_0^t f(s, x - b(t - s))ds$. Use (without proving) the last formula to derive the solution of the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) = 0, & x \in \mathbb{R}, \\ u_t(0, x) = h(x), & x \in \mathbb{R}. \end{cases}$$

where $h \in C^1(\mathbb{R})$.

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Exercise 3 **(4 + 4 + 4 points)**

a) (i) Determine the type (elliptic, hyperbolic or parabolic) of the following second order partial differential equation for a function $u : (0, \infty)^2 \to \mathbb{R}$

$$xy\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial u}{\partial x} = 0, x, y \in (0, \infty).$$

(ii) Let $U \subset \mathbb{R}^n$ open, and α a multiindex. For $u, g \in L^1_{loc}(U)$ when do we say that $\partial^\alpha u = g$ in the sense of weak derivatives? Define the Sobolev space $W^{k,p}(U)$, where $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$.

b) Consider a spherically symmetric function $\varrho : \mathbb{R}^3 \to \mathbb{R}$, which is bounded integrable and supported in the ball $B_R(0)$. Prove that for $|x| > R$ the fundamental solution γ_3 of the Laplace equation fulfills

$$\int_{\mathbb{R}^3} \gamma_3(x - y)\varrho(y)dy = \gamma_3(x) \int_{\mathbb{R}^3} \varrho(y)dy.$$

c) Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-\pi x^2}$. Prove that $\widehat{f} = f$, namely that the Fourier transformation of f is again f.

Hint: You may use without proof that $\int_{\mathbb{R}} f(x)dx = 1$.

Exercise 4 **(6 + 6 points)**

a) Let $L > 0$. Determine all functions $u : (0, \infty) \times [0, L] \to \mathbb{R}$ with

$$u_{tt} = u_{xx}\ , \qquad x \in (0, L), \quad t > 0,$$

that have the form $u(t, x) = w(t)v(x)$ and fulfill the boundary conditions

$$u\,(t, 0) = u\,(t, L) = 0,$$

for all $t > 0$.

b) Let $U \subset \mathbb{R}^n$ open and bounded and let $X := \{u \in W^{1,2}_0(U) : \|u\|_{L^2(U)} = 1\}$. We consider the functional $E : X \to \mathbb{R}$, $E(u) = \int_U |\nabla u|^2 dx$.

(i) Prove that the functional E has always a minimizer $u_0 \in X$.

(ii) State (without proving) an important theorem that we proved in the lecture using part (i).