Exam

Classical methods for partial differential equations.

Note: To pass the exam you need 16 out of 48 points.

Exercise 1  (4 + 3 + 5 points)

a) Let $U \subset \mathbb{R}^n$ be open and bounded. On the set of functions $u : U \to \mathbb{R}$ with $u \in C(U) \cap C^2(U)$, we consider an elliptic differential operator $L$ having the form

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i},$$

for some functions $a_{ij}, b^i \in C(U)$. Prove that if $Lu < 0$ in $U$, then the maximum of $u$ in $U$ is attained on $\partial U$.

b) Let $u = u(x, y)$ be a continuous function on the disc $\overline{B_R(0)} \subset \mathbb{R}^2$, which is also harmonic on $B_R(0)$. Assume that $u(R \cos(\theta), R \sin(\theta)) = 1 + 5 \cos^2(\theta)$, for $\theta \in [0, 2\pi]$. Determine the minimum of $u$ on $\overline{B_R(0)}$ and explain on how many points it is attained.

c) Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ domain and assume that $G$ is a Green’s function in $\Omega$. Prove that if $u \in C^2(\Omega)$, then for all $x \in \Omega$ we have

$$u(x) = \int_{\Omega} G(x, y)(-\Delta u(y))dy + \int_{\partial \Omega} -u(y)\nabla_y G(x, y) \cdot \nu(y)d\sigma_y.$$

Exercise 2  (6 + 6 points)

a) Let $f \in L^1(\mathbb{R}^n)$ and $\Phi$ be the fundamental solution of the heat equation in $\mathbb{R}^n$. Define $u(t, x) = \int_{\mathbb{R}^n} \Phi(t, x-y)f(y)dy$. Prove that $\lim_{t \to 0^+} \|u(t, \cdot) - f\|_{L^1(\mathbb{R}^n)} = 0$.

Hint: You may use without proof that $\lim_{h \to 0} \|\tau_h f - f\|_{L^1} = 0$, where $(\tau_h f)(x) := f(x + h)$.

b) Recall that if $f \in C^{0,1}([0, \infty) \times \mathbb{R})$ and $g \in C^1(\mathbb{R})$ then the solution of

$$\begin{align*}
  u_t + bu_x &= f(t, x), & (t, x) \in (0, \infty) \times \mathbb{R} \\
  u(0, x) &= g(x), & x \in \mathbb{R},
\end{align*}$$

(1)

is explicitly given by $u(t, x) = g(x - bt) + \int_0^t f(s, x - b(t - s))ds$. Use (without proving) the last formula to derive the solution of the initial value problem

$$\begin{align*}
  u_{tt} &= c^2 u_{xx}, & (t, x) \in (0, \infty) \times \mathbb{R}, \\
  u(0, x) &= 0, & x \in \mathbb{R}, \\
  u_t(0, x) &= h(x), & x \in \mathbb{R},
\end{align*}$$

where $h \in C^1(\mathbb{R})$.  

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Exercise 3  (4 + 4 + 4 points)

a)  (i) Determine the type (elliptic, hyperbolic or parabolic) of the following second order partial differential equation for a function $u : (0, \infty)^2 \to \mathbb{R}$

$$xy \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial x \partial y} - y \frac{\partial u}{\partial x} = 0, \quad x, y \in (0, \infty).$$

(ii) Let $U \subset \mathbb{R}^n$ open, and $\alpha$ a multiindex. For $u, g \in L^1_{\text{loc}}(U)$ when do we say that $\frac{\partial^\alpha u}{\partial x^\alpha} = g$ in the sense of weak derivatives? Define the Sobolev space $W^{k,p}(U)$, where $k \in \mathbb{N}$ and $1 \leq p \leq \infty$.

b) Consider a spherically symmetric function $\varrho : \mathbb{R}^3 \to \mathbb{R}$, which is bounded integrable and supported in the ball $B_R(0)$. Prove that for $|x| > R$ the fundamental solution $\gamma_3$ of the Laplace equation fulfills

$$\int_{\mathbb{R}^3} \gamma_3(x-y)\varrho(y)dy = \gamma_3(x)\int_{\mathbb{R}^3} \varrho(y)dy.$$ 

\textbf{c)} Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-\pi x^2}$. Prove that $\hat{f} = f$, namely that the Fourier transformation of $f$ is again $f$.

\textbf{Hint:} You may use without proof that $\int_{\mathbb{R}} f(x)dx = 1$.

Exercise 4  (6 + 6 points)

a) Let $L > 0$. Determine all functions $u : (0, \infty) \times [0, L] \to \mathbb{R}$ with

$$u_{tt} = u_{xx}, \quad x \in (0, L), \quad t > 0,$$

that have the form $u(t, x) = w(t)v(x)$ and fulfill the boundary conditions

$$u(t, 0) = u(t, L) = 0,$$

for all $t > 0$.

b) Let $U \subset \mathbb{R}^n$ open and bounded and let $X := \{u \in W^{1,2}_0(U) : \|u\|_{L^2(U)} = 1\}$. We consider the functional $E : X \to \mathbb{R}$, $E(u) = \int_U |\nabla u|^2 dx$.

(i) Prove that the functional $E$ has always a minimizer $u_0 \in X$.

(ii) State \underline{without proving} an important theorem that we proved in the lecture using part (i).