11a) Assume that the maximum is attained at $x_0 \in U$. Then $u_{x_i}(x_0) = 0$

So $Lu(x_0) = -\sum_{i,j=1}^{n} a^{ij}(x_0) u_{x_i x_j}(x_0)$

$= -\mathrm{Tr}(A(x_0) \operatorname{Hess} u(x_0))$.

where $A(x_0) = (a^{ij}(x_0))_{i,j=1}^{n}$. But since L is elliptic $A(x_0)$ is positive definite. Since u has maximum at x_0 we have $\operatorname{Hess} u(x_0)$ is positive semi-definite. Therefore,

$\mathrm{Tr}(A(x_0) \operatorname{Hess} u(x_0)) \leq 0$ so $Lu(x_0) \geq 0$

contradiction. Thus the maximum is attained on the boundary.

(1b) Since u is harmonic its minimum is attained on the boundary $\partial B_5(0)$

Since $1+5\cos^2\theta \geq 1$ and $1+5\cos^2\left(\frac{\pi}{2}\right)=1$ the minimum on $\partial B_5(0)$ and therefore on $\overline{B_5(0)}$ is 1. The minimum can not be attained on the interior because the function is not constant. On the boundary the minimum is attained for $\theta=\frac{\pi}{2}$ and for $\theta=\frac{3\pi}{2}$.

So in total the minimum is attained at 2 points.

(1c) By Green's representation formula we have that

$$u(x) = \int_\Omega \gamma_n(x-y)(-\Delta u(y))\,dy + \int_{\partial\Omega} (\gamma_n(x-y)\nabla u(y) - u(y)\nabla_y \gamma_n(x-y))\cdot \nu(y)\,d\sigma_y \quad (1)$$

Now since $G(x,y) = \gamma_n(x-y) + w(x,y)$ (3) and <u>w is harmonic w.r.t. y</u> in Ω $\forall x \in \bar{\Omega}$ we have:

$$\int_{\partial\Omega} w(x-y)\cdot \nabla u(y) - u(y)\nabla_y w(x-y)\,d\sigma_y = \int_\Omega \mathrm{div}_y (w(x-y)\cdot\nabla u(y) - u(y)\nabla_y w(x-y))$$

$$= \int_\Omega w(x-y)\,\Delta u(y)\,dy - \int_\Omega u(y)\underbrace{\Delta_y w(x-y)}_{=0}\,dy$$

$$\Rightarrow -\int w(x-y)\,\Delta u(y)\,dy + \int_{\partial\Omega} w(x-y)\nabla u(y) - u(y)\nabla_y w(x-y)\,d\sigma_y \quad (2)$$

Adding (1), (2) and using (3) we find

$$u(x) = \int G(x,y)(-\Delta u(y))\,dy + \int_{\partial\Omega} G(x,y)\nabla u(y) - u(y)\nabla_y G(x,y)\,d\sigma_y$$

since $G(x,y) = 0$ in $\Omega \times \partial\Omega$ we arrive at the desired formula

2a). We have that

$$\text{since } \int \phi(t,x-y)\,dy = 1$$

$$u(t,x) - f(x) = \int \phi(t,x-y) f(y)\,dy - \int \phi(t,x-y) f(x)\,dy$$

$$= \int \phi(t,x-y)\big(f(y) - f(x)\big)\,dy$$

$$= \int \phi(t,y)\big(f(x-y) - f(x)\big)\,dy.$$

where we have used that

$$\int \phi(t,y) f(x)\,dy = f(x) = \int \phi(t,x-y) f(x)\,dy.$$

and that $\phi(t,\cdot) * f = f * \phi(t,\cdot)$.

So $u(t,x) - f(x) = \int \frac{1}{t^{n/2}} \phi\left(1, \frac{y}{\sqrt{t}}\right)\big(f(x-y) - f(x)\big)\,dy.$

$$\xlongequal[d\tilde{y} = \frac{1}{t^{n/2}}dy]{\tilde{y} = \frac{y}{\sqrt{t}}} \int \phi(1,\tilde{y})\big(f(x - t\tilde{y}) - f(x)\big)\,dy.$$

Thus $\int |u(t,x) - f(x)|\,dx \overset{\text{Tonelli}}{=} \iint |\phi(1,\tilde{y})|\,|f(x - t\tilde{y}) - f(x)|\,d\tilde{y}\,dx$

$$\leq \iint |\phi(1,\tilde{y})| \underbrace{\int |f(x-t\tilde{y})-f(x)|\,dx}\,dy$$

$\downarrow$ due to the hint

0 as $t \to 0$ pointwise.

and $\phi(1,\tilde{y}) \int |f(x-t\tilde{y})-f(x)|\,dx \leq \underbrace{\phi(\cdot,\tilde{y})\, 2\|f\|_{L^1}}_{\in L^1}$

So we can apply dominated convergence theorem to obtain the desired result.

(2b). We illustrate how to solve it if $u(0,x)=g(x)$.

$$c^2 u_{xx} \Rightarrow \left(\partial_t^2 - c^2 \partial_x^2\right) u = 0 \Rightarrow (\partial_t - c\partial_x)\underbrace{(\partial_t + c\partial_x) u}_{v} = 0$$

$$\begin{cases} (\partial_t - c\partial_x) v = 0 \\ v(0,x) = \partial_t u(0,x) + c\partial_x u(0,x) = h(x) + c g'(x) \end{cases} \overset{(2)}{\Longrightarrow}$$

$$v(t,x) = h(x+ct) + c g'(x+ct).$$

$$\begin{cases} (\partial_t + c\partial_x) u(t,x) = h(x+ct) + c g'(x+ct) \\ u(0,x) = g(x) \end{cases} \overset{(2)}{\Longrightarrow}$$

$$u(t,x) = g(x-ct) + \int_0^t \left[h(x-c(t-s)+cs) + c g'(x-c(t-s)+cs)\right] ds$$

$$= g(x-ct) + \int_0^t h(x-ct+2cs)\,ds + \int_0^t c g'(x-ct+2cs)\,ds$$

$$= g(x-ct) + \left.\frac{g(x-ct+2cs)}{2}\right|_0^t + \int_0^t h(\overbrace{x-ct+2cs}^{y})\,ds \quad dy = 2c\,ds$$

$$=\quad \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy$$

3a) (i) The equation has the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{x^2}{2}\frac{\partial^2 u}{\partial x \partial y} + \frac{x^2}{2}\frac{\partial^2 u}{\partial y \partial x} - y\frac{\partial u}{\partial x} = 0$$

the second order terms are described by the matrix

$$A(x) = \begin{pmatrix} xy & \frac{x^2}{2} \\ \frac{x^2}{2} & 0 \end{pmatrix}.$$

Since $\det A(x) = -\frac{x^4}{4} < 0$

it follows right away that the e.v are of opposite sign so the equation hyperbolic.

b) (ii) $\partial^\alpha u = g \iff \int_U u v = (-1)^\alpha \int_U g\, \partial^\alpha v \quad \forall v \in C_c^\infty(U)$

$W^{1,p}(U) = \{u \in L^p(U) : \partial^\alpha u \in L^p(U),\ \forall |\alpha| \leq 1\}$.

(3b) We have $\int_{\mathbb{R}^3} \gamma_3(x-y)\rho(y)dy =$

due to spherical symmetry

$$= \int_{B_R(0)} \gamma_3(x-y)\rho(y)dy = \int_0^R \rho(r) \underbrace{\int_{\partial B_r(0)} \gamma_3(x-y)\,d\sigma_y}\, dr.$$

$= \gamma_3(x)|\partial B_r(0)|$ by mean value property since $|x|>r$ so $g(y)=\gamma_3(x-y)$ is harmonic in $B_r(0)$.

$$= \gamma_3(x)\int_0^R \rho(r)\,|\partial B_r(0)|\,dr = \gamma_3(x)\int_{B_R(0)} \rho(y)dy$$

$$= \gamma_3(x)\int_{\mathbb{R}^3} \rho(y)dy.$$

(3c) $f(x)=e^{-\pi x^2} \Rightarrow f'(x)=-2\pi x e^{-\pi x^2}$

$$\Rightarrow \begin{cases} f'(x) = -2\pi x f(x). \\ \text{and } f(0)=1 \end{cases}$$

Since $f \in S$ so $xf, f' \in S$ we obtain that

$$\widehat{f'}(\xi) = \widehat{-2\pi x f}(\xi) \Longrightarrow$$

$$i\hat{f}'(\xi) = \widehat{-2\pi i x f}(\xi) \Rightarrow i(2\pi i\xi)\hat{f}(\xi) = \hat{f}'(\xi)$$

$$\Rightarrow \hat{f}'(\xi) = -2\pi\xi\hat{f}(\xi).$$

But $\hat{f}(0) = \int e^{-2\pi i x \cdot 0} f(x)\,dx \overset{Hint}{=} 1.$

So $\hat{f}$ solves the same initial value problem as f so by uniqueness $\hat{f}=f$.

(4a) If we substitutue $u(t,x)=w(t)v(x)$ in $u_{tt}=u_{xx}$ we find $w''(t)v(x)=w(t)v''(x)$

$$\Rightarrow \underbrace{\frac{w''(t)}{w(t)}}_{\text{depends only on } t} = \underbrace{\frac{v''(x)}{v(x)}}_{\text{depends only on } x} = d \in \mathbb{R}$$

Thus they both have to be a constant.

So $\begin{cases} v''(x)=dv(x) \\ v(0)=0, v(L)=0 \end{cases}$ $\longrightarrow$ from boundary conditions.

<u>Case 1</u> $d>0$. Then $v(x)=c_1 e^{\sqrt{d}x}+c_2 e^{-\sqrt{d}x}$.

$v(0)=0 \Rightarrow c_1+c_2=0 \Rightarrow c_2=-c_1$

$v(L)=0 \Rightarrow c_1 e^{\sqrt{d}L}-c_1 e^{-\sqrt{d}L}=0 \Rightarrow$

$c_1 \underbrace{\left(e^{\sqrt{d}L}-e^{-\sqrt{d}L}\right)}_{\neq 0 \text{ since } d,L>0}=0 \Rightarrow c_1=0$

Therefore $c_1=c_2=0 \Rightarrow v(x)=0$ so only trivial solution.

Case 2 $d=0$. Then $v''(x)=0 \Rightarrow v(x)=c_1+c_2 x$.

$v(0)=0 \Rightarrow c_1=0$. $v(L)=0 \Rightarrow c_2 L=0 \Rightarrow c_2=0$

So $v=0$.

Case 3: $d<0$. Then $v(x)=c_1\cos(\sqrt{-d}\,x)+c_2\sin(\sqrt{-d}\,x)$

$v(0)=0 \Rightarrow c_1=0$. So $v(x)=c_2\sin(\sqrt{-d}\,x)$.

$$v(L)=0 \Rightarrow \underbrace{c_2}_{\text{is nonzero}}\ \underbrace{\sin(\sqrt{-d}\,L)}_{\text{only when this is zero.}}=0.$$

So nontrivial solution $\rightsquigarrow \sin(\sqrt{-d}\,L)=0$.

$$\sqrt{-d}\,L = n\pi,\ n\in\mathbb{N} \qquad d_n=-\frac{n^2\pi^2}{L^2}.$$

$$\rightsquigarrow v_n(x)=\sin\left(\frac{n\pi}{L}x\right).$$

In this case we find $w_n''(t)=d_n w_n(t)$.

$$\rightsquigarrow w_n(t)=c_1\cos\left(\frac{n\pi x}{L}\right)+c_2\sin\left(\frac{n\pi x}{L}\right).$$ 12:48

So we end up with solutions of the form

$$u_n(x,t)=\left(c_1\cos\left(\frac{n\pi x}{L}\right)+c_2\sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{n\pi}{L}x\right),\ n\in\mathbb{N}.$$

(4b) (i) Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a minimizing sequence. Then $\|u_n\|^2_{W^{1,2}(U)} = \underbrace{\|u_n\|^2}_{=1 \text{ since } u_n\in X} + E(u_n)$

$\downarrow$ bounded since u_n minimizing sequence

So $\|u_n\|_{W_0^{1,2}(U)}$ is bounded thus

$\exists\, u_0 \in W_0^{1,2}(U)$ and a subsequence u_{n_k} of u_n sucht that $u_{n_k} \rightharpoonup u_0$ in $W_0^{1,2}(U)$.

By Rellich Kondrachov it follows that

$u_{n_k} \to u_0$ in $L^2(U)$ so $\|u_0\|_{L^2} = 1$

Thus $u_0 \in X$. On the other hand

$$\|u_0\|^2_{W_0^{1,2}(U)} \le \liminf_{k\to\infty} \|u_{n_k}\|^2_{W_0^{1,2}(U)} \Rightarrow$$

$$\|u_0\|_{L^2(U)} + E(u_0) = \underbrace{\|u_{n_k}\|^2}_{=1} + \liminf_{k\to\infty} E(u_{n_k})$$

$\Rightarrow E(u_0) \le \liminf\limits_{k\to\infty} E(u_{n_k})$ and since u_{n_k} is minimizing and $u_0\in X$ it follows that u_0 is a minimizer

(ii) We used part (i) to prove that there is an orthonormal basis of $L^2(U)$ consisting of functions $(u_n)_{n \in \mathbb{N}} \subset W_0^{1,2}(U)$ that are eigenfunctions of the Laplacian.