Fundamental solution of the Laplace equation in $\mathbb{R}^n$ $n \geqslant 2$: $\gamma_n : \mathbb{R}^n/\{0\} \to \mathbb{R}$ given by

$$\gamma_n(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & \text{if } n = 2 \\ \frac{1}{(n-2)|\partial B_1(0)|} |x|^{2-n}, & \text{if } n \geqslant 3, \end{cases} \tag{1}$$

Theorem 0.1 (Green's Representation formula)**.** *Assume that $\Omega \subset R^n$ is a C^1 domain and let $u \in C^2(\overline{\Omega})$. If γ_n is the fundamental solution of the Laplace equation then for all $x \in \Omega$ we have that*

$$u(x) = \int_\Omega \gamma_n(x-y)(-\Delta u(y))dy + \int_{\partial\Omega} \Big(\gamma_n(x-y)\nabla u(y) - u(y)\nabla_y \gamma_n(x-y)\Big) \cdot \nu(y) d\sigma_y,$$

where $\nu(y)$ is the unit outward normal vector of $\partial\Omega$ at y.

Theorem 0.2 (Green's function for the unit ball)**.** *The Green's function for the ball $B_1(0)$ exists for $n \geqslant 2$ and it is given as follows (for $x \neq 0$ we define $x^* := \frac{x}{|x|^2}$):*

$$G(x,y) = \begin{cases} \gamma_n(y-x) - \gamma_n(|x|(x^* - y)) & \text{if } x \neq 0 \\ \gamma_n(y) - \frac{1-\delta_{n2}}{(2-n)|\partial B_1(0)|}, & \text{if } x = 0, \end{cases}$$

Theorem 0.3. *Let $u \in C^2(\overline{B_1(0)})$ be a harmonic function with $u = f \in C(\partial B_1(0))$ on $\partial B_1(0)$. Then*

$$u(x) = \frac{1 - |x|^2}{|\partial B_1(0)|} \int_{\partial B_1(0)} \frac{f(y)}{|x-y|^n} d\sigma_y.$$

Definition 0.4. *Fundamental solution of $u_t(t,x) = \Delta_x u(t,x)$ in $(0,\infty) \times \mathbb{R}^n$: $\Phi(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$.*

For all multiindices α and $b, d \in \mathbb{R}$ with $0 < b < d$ there exists $C_{\alpha,b}$ such that for all $t \in [b,d]$

$$|\partial_\alpha \Phi(t,x)| \leqslant C_{\alpha,b} e^{-\frac{|x|^2}{8d}}$$

Fourier series Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic function, $f|_{[-\pi,\pi)}$ integrierbar. Fourier coefficients

$$a_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos\left(\frac{2\pi}{T} kt\right) dt, \ k \in \mathbb{N} \cup \{0\}, \quad b_k = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin\left(\frac{2\pi}{T} kt\right) dt, k \in \mathbb{N},$$

Fourier series of f: $\quad \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos\left(\frac{2\pi}{T} kt\right) + \sum_{k=1}^\infty b_k \sin\left(\frac{2\pi}{T} kt\right)$.

0.1 The wave equation in $\mathbb{R}$

We consider the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \\ u_t(0,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Theorem 0.5. *If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, then the initial value problem (3) has a unique solution in $C^2([0,\infty) \times \mathbb{R})$, which is explicitly given by*

$$u(t,x) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) dy.$$

Example 0.6. *Let $g \in C^2([0,\infty)), h \in C^1([0,\infty))$. Suppose that $g(0) = h(0) = 0$ and $g''(0) = 0$. Then the boundary initial value problem*

$$\begin{cases} u_{tt} = u_{xx}, \quad (t,x) \in [0,\infty) \times (0,\infty), \\ u(t,0) = 0,\ t \geqslant 0, \qquad u(0,x) = g(x),\ x \geqslant 0, \qquad u_t(0,x) = h(x),\ x \geqslant 0. \end{cases} \tag{3}$$

has a unique solution in $C^2([0,\infty) \times [0,\infty))$. This solution is given by

$$u(t,x) = \begin{cases} \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy, & \text{if } x \geqslant t \geqslant 0, \\ \frac{g(x+t)-g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{t+x} h(y)dy, & \text{if } t \geqslant x \geqslant 0. \end{cases} \tag{4}$$

Consider the initial value problem
$$\begin{cases} u_{tt} = \Delta u, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = g(x), \quad x \in \mathbb{R}^n, \\ u_t(0,x) = h(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{5}$$

Corollary 0.7. *Let $u \in C^2([0,\infty) \times \mathbb{R}^3)$ be a solution of (5). Fix $x \in \mathbb{R}^3$. Then $\widetilde{U} = rU$ solves*

$$\begin{cases} \widetilde{U}_{tt} = \widetilde{U}_{rr}, \quad (t,r) \in [0,\infty) \times [0,\infty), \\ \widetilde{U}(0,x,r) = \widetilde{G}(x,r), \quad r \in [0,\infty), \\ \widetilde{U}_t(0,x,r) = \widetilde{H}(x,r), \quad r \in [0,\infty), \end{cases}$$

where $\widetilde{G} = rG, \widetilde{H} = rH$, with $U(t,x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(t,y)d\sigma_y$, $G(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} g(y)d\sigma_y$, $H(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} h(y)d\sigma_y$.

Case $n = 3$ Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Formula for the solution of (5)

$$u(t,x) := \frac{1}{4\pi t^2} \int_{\partial B_t(x)} (th(y) + g(y) + \nabla g(y)(y-x))d\sigma_y. \tag{6}$$

Case $n = 2$ Let $g \in C^3(\mathbb{R}^2)$ and $f \in C^2(\mathbb{R}^2)$. Formula for the solution of (5)

$$u(t,x) := \frac{1}{2\pi t} \int_{B_t(x)} \frac{(th(y) + g(y) + \nabla g(y)(y-x))}{\sqrt{t^2 - |x-y|^2}} dy. \tag{7}$$

Definition 0.8 (Fourier transformation). *If $f \in L^1(\mathbb{R}^n)$ we define $\widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x)dx$.*

Theorem 0.9. *(1) If $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \widehat{f}\widehat{g}$, where recall that $f * g(x) = \int f(x-y)g(y)dy$.*
(2) If $f \in L^1$ and $\tau_a f(x) := f(x+a)$ and $U_a f(x) = e^{2\pi i a \cdot x} f(x)$, then $\widehat{\tau_a f} = U_a \widehat{f}$, $\widehat{U_a f} = \tau_{-a} \widehat{f}$.
(3) If T is an invertible linear transformation in $\mathbb{R}^n$, then $\widehat{f \circ T}(\xi) = \frac{1}{|\det T|} \widehat{f}((T^{-1})^* \xi)$.

More properties of the Fourier transform

Theorem 0.10. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and β a multi-index. We define $(M_\beta f)(x) := (2\pi i x)^\beta f(x)$. Then*
(1) $\widehat{\partial^\beta f} = M_\beta \widehat{f}$ (2) $\widehat{f} \in C^\infty$ and $\partial^\beta \widehat{f} = (-1)^{|\beta|} \widehat{M_\beta f}$. (3) $\widehat{f} \in \mathcal{S}$.

Theorem 0.11. *Let $f(x) = e^{-\pi a |x|^2}$, where $a > 0$, then $\widehat{f}(\xi) = a^{-n/2} e^{-\pi |\xi|^2 / a}$.*

Theorem 0.12. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\int f \widehat{g} = \int \widehat{f} g$.*

Theorem 0.13 (Fourier inversion formula). *If $f \in \mathcal{S}$, then $f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$.*

For $f \in L^1(\mathbb{R}^n)$ we define $\check{f}(x) = \int f(\xi) e^{2\pi i x \cdot \xi} d\xi = \widehat{f}(-x)$.

Corollary 0.14. *$\check{\widehat{f}} = f = \widehat{\check{f}}$. Moreover, the Fourier transform is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to itself.*

Theorem 0.15 (Plancherel formula). *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$.*

Theorem 0.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. Suppose that $u \in C([0,T] \times \overline{\Omega}) \cap C^{1,2}((0,T) \times \Omega)$ with $u_t = \Delta_x u$ in $(0,T) \times \Omega$. Then u attains its maximum either on $\{0\} \times \Omega$ or on $[0,T] \times \partial\Omega$.*