The first lecture is going to be introductory and general. We will briefly explain how partial differential equations arise in applications and we will give some examples of partial differential equations. We will derive the heat equation $u_t = \Delta u$ and the Poisson equation $-\Delta u = f$. Time permitting we will begin with the discussion properties of the Poisson equation and the Laplace equation $-\Delta u = 0$.

1 The Laplace and Poisson equation

1.1 Harmonic functions and maximum/minimum Principle

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be open. A function $u \in C^2(\Omega)$ is called a harmonic function in Ω if $\Delta u := \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} = 0$ in Ω.*

For $R > 0$ and $x_0 \in \mathbb{R}^n$ recall that

$$B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}, \quad \partial B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| = R\}$$

$$|B_R(x_0)| := \int_{B_R(x_0)} 1dx \text{ volume of } B_R(x_0), \quad |\partial B_R(x_0)| := \int_{\partial B_R(x_0)} 1d\sigma_x \text{ area of } \partial B_R(x_0).$$

Definition 1.2. *Gaussian mean value: Let $R > 0, x_0 \in R^n$ and $u : \overline{B_R(x_0)} \to R$ continuous. Then the integrals*

$$m_R(x_0, u) := \frac{1}{|\partial B_R(x_0)|} \int_{\partial B_R(x_0)} u(x) d\sigma_x,$$
$$M_R(x_0, u) := \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) dx,$$

are called the average value of u on the surface ball, respectively the average value of u on the interior ball.

Theorem 1.3 (Mean value Property for harmonic functions)**.** *If $u : \overline{B_R(x_0)} \to \mathbb{R}$ is continuous and harmonic in $B_R(x_0)$ then $u(x_0) = m_R(x_0, u) = M_R(x_0, u)$.*

Definition 1.4. *An $\Omega \subset \mathbb{R}^n$ is called a domain if it is open and connected.*

Theorem 1.5 (Maximum and minimum Principle for harmonic functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u : \Omega \to \mathbb{R}$ harmonic and not constant. Then u has no maximum or minimum in Ω.*

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C(\overline{\Omega})$ be a harmonic function in Ω. Let $M = \max\{u(x) : x \in \overline{\Omega}\}, m = \min\{u(x) : x \in \overline{\Omega}\}$. Then*
(i) $M = \max\{u(x) : x \in \partial\Omega\}, m = \min\{u(x) : x \in \partial\Omega\}$.
(ii) If $u = c \in \mathbb{R}$ on $\partial\Omega$ then $u = c$ in Ω.

Corollary 1.7. *Let Ω be a bounded domain, $g \in C(\Omega), f \in C(\partial\Omega)$. Then there is at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ solving the Poisson equation $-\Delta u = g$ in Ω with the boundary condition $u = f$ on $\partial\Omega$.*

1.2 Fundamental solution of the Laplace equation

Definition 1.8 (Fundamental solution of the Laplace equation)**.** *Let $n \in \mathbb{N}$ with $n \geqslant 2$. The function $\gamma_n : \mathbb{R}^n/\{0\} \to \mathbb{R}$ given by*

$$\gamma_n(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & \text{if } n = 2 \\ \frac{1}{(n-2)\omega_n} |x|^{2-n}, & \text{if } n \geqslant 3, \end{cases} \tag{1}$$

where $\omega_n = |\partial B_1(0)|$, is called fundamental solution of the Laplace equation in $\mathbb{R}^n$.

Theorem 1.9 (A property of the Fundamental solution of the Laplace equation)**.** *Suppose that* $f \in C^2(\mathbb{R}^n)$ *has compact support. Then*
(i) $\int \gamma_n(x)(-\Delta f(x))dx = f(0)$.
(ii) The function $u(x) = \gamma_n * f := \int \gamma_n(x-y)f(y)dy$ *is a solution of the Poisson equation* $-\Delta u = f$.

Definition 1.10 (C^k/Lipschitz domains)**.** *Let* $k \in \mathbb{N}$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. We say that* Ω *is* C^k*, respectively Lipschitz at* $x_0 \in \partial\Omega$ *if there exists* $r > 0$ *and a* C^k*, respectively Lipschitz function* $\gamma : \mathbb{R}^n \to \mathbb{R}$ *such that -upon relabeling and reorienting the coordinate axes if necessary- we have*

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \quad | \quad x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

If Ω *is* C^k*, respectively Lipschitz at all its boundary points then it is called a* C^k*, respectively Lipschitz domain.*

Theorem 1.11 (Green's Representation formula)**.** *Assume that* $\Omega \subset R^n$ *is a* C^1 *domain and let* $u \in C^2(\overline{\Omega})$*. If* γ_n *is the fundamental solution of the Laplace equation then for all* $x \in \Omega$ *we have that*

$$u(x) = \int_{\Omega} \gamma_n(x-y)(-\Delta u(y))dy + \int_{\partial\Omega} \Big(\gamma_n(x-y)\nabla u(y) - u(y)\nabla_y \gamma_n(x-y)\Big) \cdot \nu(y) d\sigma_y,$$

where $\nu(y)$ *is the unit outward normal vector of* $\partial\Omega$ *at* y*.*

1.3 Green's functions on bounded domains

Definition 1.12 (Green's function)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a domain. A function* $G(x,y) := \gamma_n(x-y) + w(x,y), x \neq y, x \in \Omega, y \in \overline{\Omega}$ *is called a Green function for the Laplace operator in* Ω *if:*
(i) $\forall x \in \Omega, w(x,.) \in C^2(\overline{\Omega})$ *and* $\Delta_y w(x,y) = 0$ *in* $\overline{\Omega}$*.*
(ii) $\forall x \in \Omega, y \in \partial\Omega$ *we have that* $G(x,y) = 0$*.*

Example 1.13 (An important one)**.** *Let* $\Omega = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$*. Then the function* $G(x,y) = \gamma_n(x-y) + \gamma_n(x-y^*)$ *is a Green's function for* Ω*, where for* $y = (y_1, ..., y_{n-1}, y_n)$ *we define* $y^* = (y_1, ..., y_{n-1}, -y_n)$*.*

Theorem 1.14 (Corollary of Theorem 1.11)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a* C^1 *domain and assume that* G *is a Green's function in* Ω*. If* $u \in C^2(\overline{\Omega})$*, then for all* $x \in \Omega$ *we have*

$$u(x) = \int_{\Omega} G(x,y)(-\Delta u(x)) + \int_{\partial\Omega} -u(y)\nabla_y G(x,y) \cdot \nu(y) d\sigma_y.$$

Theorem 1.15 (Green's function for the unit ball)**.** *The Green's function for the ball* $B_1(0)$ *exists for* $n \geqslant 2$ *and it is given as follows:*

$$G(x,y) = \begin{cases} \gamma_n(y-x) - \gamma_n(|x|(x^* - y)) & \text{if } x \neq 0 \\ \gamma_n(y) - \frac{1-\delta_{n2}}{(2-n)\omega_n}, & \text{if } x = 0, \end{cases}$$

where for $x \neq 0$ *we defined* $x^* := \frac{x}{|x|^2}$*.*

Theorem 1.16. *Let* $u \in C^2(\overline{B_1(0)})$ *be a harmonic function with with* $u = f \in C(\partial B_1(0))$ *on* $\partial B_1(0)$*. Then*

$$u(x) = \frac{1-|x|^2}{\omega_n} \int_{\partial B_1(0)} \frac{f(y)}{|x-y|^n} d\sigma_y.$$

Theorem 1.17. *Let* $f \in C(\partial B_1(0))$ *and*

$$u(x) = \begin{cases} \frac{1-|x|^2}{\omega_n} \int_{\partial B_1(0)} \frac{f(y)}{|x-y|^n} d\sigma_y, & \text{if } x \in B_1(0) \\ f(x), & \text{if } x \in \partial B_1(0). \end{cases}$$

Then $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)})$ *and* u *is harmonic in* $B_1(0)$ *(*u *is called harmonic extension of* f *at* 0*).*

2 Elliptic operators and Maximum Principles

2.1 Elliptic operators

Let $U \subset R^n$ be open and bounded. On the set of functions $u : U \to \mathbb{R}$ be in $C^2(U)$, we consider a differential operator L having the form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum b^i(x)u_{x_i} + c(x)u. \tag{2}$$

for some coefficient functions a^{ij}, b^i, c. Our aim is to study the boundary value problem

$$\begin{cases} Lu = f \text{ on } U \\ u = 0 \text{ on } \partial U. \end{cases} \tag{3}$$

We assume from now on, that

$$a^{ij}, b^i, c \in L^\infty(U) \cap C(U). \tag{4}$$

.

Definition 2.1. *We say that the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2 \tag{5}$$

for almost all $x \in U$ and all $\xi \in \mathbb{R}^n$.

This section discusses conditions under which solutions of elliptic boundary value problems attain their minimum or maximum at the boundary. Below L is the operator given by (2). We will always assume that a^{ij}, b^i, c are continuous. Without loss of generality we also assume the symmetry condition $a^{ij} = a^{ji}$, $i, j = 1, \ldots, n$. We also assume that U is open and bounded.

2.2 Weak maximum principles

We will first prove the following preliminary lemma:

Lemma 2.2. *If two matrices $A, B \in R^{n\times n}$ are symmetric and positive definite then $Tr(AB) \geqslant 0$.*

Theorem 2.3 (Weak maximum principle)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U.*
(i) If $Lu \leqslant 0$ in U then $\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x)$.
(ii) If $Lu \geqslant 0$ in U, then $\min_{x\in\overline{U}} u(x) = \min_{x\in\partial U} u(x)$.

The following theorem is a generalization of the previous when $c \geqslant 0$. We define $u^+ = max(u, 0)$ and $u^- = -min(u, 0)$.

Theorem 2.4 (Weak maximum principle for $c \geqslant 0$)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c \geqslant 0$ in U.*
(i) If $Lu \leqslant 0$ in U then $\max_{x\in\overline{U}} u(x) \leqslant \max_{x\in\partial U} u^+(x)$.
(ii) If $Lu \geqslant 0$ in U, then $\min_{x\in\overline{U}} u(x) \geqslant -\max_{x\in\partial U} u^-(x)$.
In particular if $Lu = 0$ in U, then $\max_{x\in\overline{U}} |u(x)| = \max_{x\in\partial U} |u(x)|$.

2.3 Hopf's lemma

The following lemma has the goal to strengthen the weak maximum principles.

Definition 2.5. *Let $x^0 \in \partial U$. We say that U satisfies the interior ball condition at x^0 if there is an open ball $B \subset U$ such that $x^0 \in \partial B$.*

Lemma 2.6 (Hopf's Lemma)**.** *Assume that $u \in C^2(U) \cap C^1(\overline{U})$. Suppose that $Lu \leqslant 0$ in U, and that there exists a point $x^0 \in \partial U$ such that*

$$u(x^0) > u(x), \quad \forall x \in U.$$

Assume that U satisfies the interior ball condition at x^0. Then

(i) If $c = 0$ in U then $\frac{\partial u}{\partial n}(x^0) > 0$, where n is the outer unit normal to B.

(ii) If $c \geqslant 0$ in U and $u(x^0) \geqslant 0$ then the same conclusion holds.

2.4 Strong maximum principles

Theorem 2.7 (Strong maximum principle)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U. Suppose also that U is connected open and bounded.*

(i) If $Lu \leqslant 0$ in U and u attains its maximum over $\overline{U}$ at an interior point, then u is constant within U.

(ii) If $Lu \geqslant 0$ in U and u attains its minimum over $\overline{U}$ at an interior point, then u is constant within U.

Theorem 2.8 (Strong maximum principle for $c \geqslant 0$)**.** *Assume that $u \in C^2(U) \cap C(\overline{U})$ and $c \geqslant 0$ in U. Suppose also that U is connected open and bounded.*

(i) If $Lu \leqslant 0$ in U and u attains a nonnegative maximum over $\overline{U}$ at an interior point, then u is constant within U.

(ii) If $Lu \geqslant 0$ in U and u attains a nonpositive minimum over $\overline{U}$ at an interior point, then u is constant within U.

3 The heat equation in $\mathbb{R}^n$

In this section we will study the initial value problem

$$u_t(t,x) = \Delta_x u(t,x) \text{ in } (0,\infty) \times \mathbb{R}^n \tag{6}$$

$$u(0,x) = g(x) \text{ on } \mathbb{R}^n. \tag{7}$$

as well as

$$u_t(t,x) = \Delta_x u(t,x) + f(t,x) \text{ in } (0,\infty) \times \mathbb{R}^n \tag{8}$$

$$u(0,x) = g(x) \text{ on } \mathbb{R}^n. \tag{9}$$

This is the homogeneous respectively inhomogeneous heat equation on R^n with initial condition g and with source f.

3.1 The homogeneous case

Our aim is here to find a solution of (6) with $f = 0$. We will first find the fundamental solution of the heat equation namely the solution of (6) for $g = \delta$. To this end the following lemma will be useful:

Lemma 3.1. *If $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a solution of (6) then*
(i) $u_{\lambda,\mu}(t,x) := \mu u(\lambda^2 t, \lambda x)$ is a solution of (6) as well $\forall \lambda > 0, \mu \in \mathbb{R}$.
(ii) If moreover $u(t,.) \in L^1(\mathbb{R}^n)$ for all $t > 0$ then for all $\lambda > 0$, $u_\lambda(t,x) := \lambda^n u(\lambda^2 t, \lambda x)$ is a solution of (6) with $\int u_\lambda(t,x)dx = \int u(\lambda^2 t, x)dx, \forall t > 0$.

Definition 3.2. *A solution of (6) is called self-similar if $u_\lambda = u$ for all $\lambda > 0$.*

Lemma 3.3. *Let $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be differentiable with $u_\lambda = u$ for all $\lambda > 0$ and with $u(t,x) = v(t,|x|)$ for some function $v : (0,\infty) \times [0,\infty) \to \mathbb{R}$. Then u is a solution of (6) if and only if there exists $c \in \mathbb{R}$ such that*

$$u(t,x) = \frac{c}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

In this case we have that $\int_{R^n} u(t,x)dx = c(4\pi)^{\frac{n}{2}}$.

Definition 3.4. *We define the fundamental solution of (6) by*

$$\Phi(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

From lemma 3.3 it follows that $\Phi(t,x)$ is a solution of the heat equation and that $\int_{\mathbb{R}^n} \Phi(t,x)dx = 1$ for all $t > 0$.

Lemma 3.5. *If $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then*

$$\lim_{t \to 0+} \int_{\mathbb{R}^n} \Phi(t,x)g(x)dx = g(0).$$

Theorem 3.6. *Assume that $g \in C(R^n) \cap L^\infty(\mathbb{R}^n)$. Then the function $u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y)g(y)dy$ is a solution of (6)-(7). Moreover $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$.*

3.2 The inhomogeneous case $f \neq 0$

A function $u : (0,\infty) \times \mathbb{R}^n$ $(t,x) \to u(t,x)$ is defined to be in $C^{1,2}((0,\infty) \times \mathbb{R}^n)$ if it is continuously differentiable with respect to t and if it is twice continuously differentiable with respect to x

Theorem 3.7. *Assume that $f \in C^{1,2}((0,\infty) \times \mathbb{R}^n) \cap C([0,\infty) \times \mathbb{R}^n)$ is bounded and has bounded derivatives. Let*

$$u(t,x) := \int_0^t \int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y) dy ds. \tag{10}$$

Then $u \in C^{1,2}((0,\infty) \times \mathbb{R}^n)$ is a solution of (8) with

$$\lim_{(x,t) \to (x_0,0), t>0} u(t,x) = 0, \forall x_0 \in \mathbb{R}^n.$$

Corollary 3.8. *Under the same assumptions for g as in Theorem 3.6 and for f as in Theorem 3.7 the function*

$$u(t,x) := \int_{\mathbb{R}^n} \Phi(t, x-y)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y) dy ds. \tag{11}$$

is in $C^{1,2}((0,\infty) \times \mathbb{R}^n)$ and is a solution of (8) with

$$\lim_{(t,x) \to (0,x_0), t>0} u(t,x) = g(x_0), \forall x_0 \in \mathbb{R}^n.$$

4 Separation of variables

Reminder of Fourier series Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic function, $f|_{[-\pi,\pi)}$ integrierbar. the numbers a_k, b_k

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt,\ k \in \mathbb{N} \cup \{0\}, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt, k \in \mathbb{N},$$

heißen <u>Fourier coefficients</u> of f und the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kt) + \sum_{k=1}^{\infty} b_k\sin(kt),$$

is called <u>Fourier series</u> of f. For any $N \in \mathbb{N}$ we define $S_N(f) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k\cos(kt) + \sum_{k=1}^{N} b_k\sin(kt)$.

Theorem 4.1. *Assume that $f : \mathbb{R} \to \mathbb{C}$ is C^2 and 2π -periodic. Then $S_N(f) \to f$ uniformly as $N \to \infty$.*

Theorem 4.2. *Assume that $f : \mathbb{R} \to \mathbb{C}$ and 2π -periodic and integrable. Then $\|S_N(f) - f\|_{L^2([-\pi,\pi])} \to 0$ as $N \to \infty$.*

The following examples are application of the ansatz of separation of variables. The assumptions are not optimal, they are made for simplicity of the proofs and for illustration of the method. We will also demonstrate how the solutions are computed.

Example 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^2, odd, 2π-periodic with $f(0) = f(\pi) = 0$. Then the problem*

$$\begin{cases} u_t = u_{xx}, \quad (t,x) \in (0,\infty) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, \\ u(0,x) = f(x), \quad x \in [0,\pi], \end{cases} \tag{12}$$

has a unique solution in $C^2((0,\infty)\times(0,\pi)) \cap C^2([0,\infty)\times[0,\pi])$. This solution is in fact in $C^\infty((0,\infty)\times[0,\pi])$.

Example 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^4, odd, 2π-periodic with $f(0) = f(\pi) = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ have the same properties except being C^3 instead of C^4. Then the problem*

$$\begin{cases} u_{tt} = u_{xx}, \quad (t,x) \in (0,\infty) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, \\ u(0,x) = f(x), \quad x \in [0,\pi], \\ u_t(0,x) = g(x), \quad x \in [0,\pi], \end{cases} \tag{13}$$

has a unique solution in $C^2([0,\infty) \times [0,\pi])$.

5 The transport and wave equation in $\mathbb{R}^n$

5.1 The (linear) transport equation in $\mathbb{R}^n$

We consider the initial value problem

$$\begin{cases} u_t + b \cdot \nabla_x u = f(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{14}$$

$b \in \mathbb{R}^n$, $f : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are given and $u \in C^1([0,\infty) \times \mathbb{R}^n)$ is unknown.

Theorem 5.1. *If $f = 0$ and $g \in C^1(\mathbb{R}^n)$ then the initial value problem* (14) *has a unique solution in $C^1([0,\infty) \times \mathbb{R}^n)$. This solution is explicitly given by $u(t,x) = g(x - bt)$.*

Theorem 5.2. *If $f \in C^{0,1}([0,\infty) \times \mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$ then the initial value problem* (14) *has a unique solution in $C^1([0,\infty) \times \mathbb{R}^n)$. This solution is explicitly given by*

$$u(t,x) = g(x - bt) + \int_0^t f(s, x - b(t-s))ds.$$

5.2 The wave equation in $\mathbb{R}$

We consider the initial value problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x), & x \in \mathbb{R}, \\ u_t(0,x) = h(x), & x \in \mathbb{R}. \end{cases} \tag{15}$$

Theorem 5.3. *If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, then the initial value problem* (15) *has a unique solution in $C^2([0,\infty) \times \mathbb{R})$, which is explicitly given by*

$$u(t,x) = \frac{g(x+ct) + g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.$$

Example 5.4. *Let $g \in C^2([0,\infty)), h \in C^1([0,\infty))$. Suppose that $g(0) = h(0) = 0$ and $g''(0) = 0$. Then the boundary initial value problem*

$$\begin{cases} u_{tt} = u_{xx}, & (t,x) \in [0,\infty) \times (0,\infty), \\ u(t,0) = 0, & t \geqslant 0, \\ u(0,x) = g(x), & x \geqslant 0, \\ u_t(0,x) = h(x), & x \geqslant 0. \end{cases} \tag{16}$$

has a unique solution in $C^2([0,\infty) \times [0,\infty))$. This solution is given by

$$u(t,x) = \begin{cases} \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy, & \text{if } x \geqslant t \geqslant 0, \\ \frac{g(x+t)-g(t-x)}{2} + \frac{1}{2}\int_{t-x}^{t+x} h(y)dy, & \text{if } t \geqslant x \geqslant 0. \end{cases} \tag{17}$$

5.3 The wave equation in R^n, $n \geqslant 2$

We consider the initial value problem

$$\begin{cases} u_{tt} = \Delta u, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = h(x), & x \in \mathbb{R}^n. \end{cases} \tag{18}$$

Spherical averages Let u, g, h be as in (18). We define

$$U(t,x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(t,y) d\sigma_y,$$

$$F(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y) d\sigma_y,$$

$$H(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} h(y) d\sigma_y.$$

Theorem 5.5. *Let $u \in C^2([0,\infty) \times \mathbb{R}^n)$ be a solution of* (18)*. Let $x \in \mathbb{R}^n$. Then $U(.,x,.)$ is a solution of the Euler-Poisson-Darboux equation*

$$\begin{cases} U_{tt} = U_{rr} + \frac{n-1}{r} U_r, & (t,r) \in [0,\infty) \times [0,\infty), \\ U(0,x,r) = G(x,r), & r \in [0,\infty), \\ U_t(0,x,r) = H(x,r), & r \in [0,\infty). \end{cases}$$

The case $n = 3$

Corollary 5.6. *Let $u \in C^2([0,\infty) \times \mathbb{R}^3)$ be a solution of* (18)*. Fix $x \in \mathbb{R}^3$. Then $\widetilde{U} = rU$ solves*

$$\begin{cases} \widetilde{U}_{tt} = \widetilde{U}_{rr}, & (t,r) \in [0,\infty) \times [0,\infty), \\ \widetilde{U}(0,x,r) = \widetilde{G}(x,r), & r \in [0,\infty), \\ \widetilde{U}_t(0,x,r) = \widetilde{H}(x,r), & r \in [0,\infty), \end{cases}$$

where $\widetilde{G} = rG, \widetilde{H} = rH$.

Theorem 5.7. *Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Define on $(0,\infty) \times \mathbb{R}^3$*

$$u(t,x) := \frac{1}{4\pi t^2} \int_{\partial B_t(x)} (th(y) + g(y) + \nabla g(y)(y-x)) d\sigma_y. \tag{19}$$

Then $u \in C^2((0,\infty) \times \mathbb{R}^3)$, $u_{tt} = \Delta u$. Moreover

$$\lim_{(t,x)\to(0,x_0),t>0} u(t,x) = g(x_0), \lim_{(t,x)\to(0,x_0),t>0} u_t(t,x) = h(x_0), \forall x_0 \in \mathbb{R}^3.$$

Furthermore, every solution of (18) *(for $n = 3$) has the form* (19)*.*

The case $n = 2$

Theorem 5.8. *Let $g \in C^3(\mathbb{R}^2)$ and $f \in C^2(\mathbb{R}^2)$. Define on $(0,\infty) \times \mathbb{R}^2$*

$$u(t,x) := \frac{1}{2\pi t} \int_{B_t(x)} \frac{(th(y) + g(y) + \nabla g(y)(y-x))}{\sqrt{t^2 - |x-y|^2}} dy. \tag{20}$$

Then $u \in C^2((0,\infty) \times \mathbb{R}^2)$, and $u_{tt} = \Delta u$. Moreover

$$\lim_{(t,x)\to(0,x_0),t>0} u(t,x) = g(x_0), \lim_{(t,x)\to(0,x_0),t>0} u_t(t,x) = h(x_0), \forall x_0 \in \mathbb{R}^2.$$

Furthermore, every solution of (18) *(for $n = 2$) has the form* (20)*.*

6 Fourier transform

Definition and some properties of the Fourier transform

Definition 6.1 (Fourier transformation)**.** *If $f \in L^1(\mathbb{R}^n)$ we define*

$$\widehat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx.$$

Theorem 6.2. *(1) If $f, g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \widehat{f}\widehat{g}$, where recall that $f * g(x) = \int f(x-y)g(y)dy$.*
(2) If $f \in L^1$ and $\tau_a f(x) := f(x+a)$ and $U_a f(x) = e^{2\pi i a \cdot x} f(x)$, then $\widehat{\tau_a f} = U_a \widehat{f}$, $\widehat{U_a f} = \tau_{-a} \widehat{f}$.
(3) If T is an invertible linear transformation in $\mathbb{R}^n$, then $\widehat{f \circ T}(\xi) = \frac{1}{|\det T|} \widehat{f}((T^{-1})^* \xi)$.

The class of Schwartz functions

An n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers will be called a multiindex. We define $|\alpha| = \sum_{j=1}^n \alpha_j$, and for $x = (x_1, ...x_n) \in \mathbb{R}^n$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}}.$$

Definition 6.3. *We define the vector space of Schwartz functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ as*

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty, \text{ for all multiindices } \alpha, \beta\}.$$

More properties of the Fourier transform

Theorem 6.4. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and β a multi-index. We define $(M_\beta f)(x) := (2\pi i x)^\beta f(x)$. Then*
(1) $\widehat{\partial^\beta f} = M_\beta \hat{f}$
(2) $\hat{f} \in C^\infty$ and $\partial^\beta \hat{f} = (-1)^{|\beta|} \widehat{M_\beta f}$, where.
(3) $\hat{f} \in \mathcal{S}$.

Theorem 6.5 (Riemann-Lebesgue Lemma)**.** *If $f \in L^1$ then $\hat{f}$ is continuous and converges to zero at infinity.*

Theorem 6.6. *Let $f(x) = e^{-\pi a |x|^2}$, where $a > 0$, then $\hat{f}(\xi) = a^{-n/2} e^{-\pi |\xi|^2 / a}$.*

Theorem 6.7. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\int f\hat{g} = \int \hat{f} g$.*

Theorem 6.8 (Fourier inversion formula)**.** *If $f \in \mathcal{S}$, then $f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$.*

For $f \in L^1(\mathbb{R}^n)$ we define $\check{f}(x) = \int f(\xi) e^{2\pi i x \cdot \xi} d\xi = \hat{f}(-x)$.

Corollary 6.9. *$\check{\hat{f}} = f = \hat{\check{f}}$. Moreover, the Fourier transform is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to itself.*

Theorem 6.10 (Plancherel Theorem)**.** *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$. Moreover, the Fourier transformation extends uniquely to a unitary isomorphism of L^2 onto itself.*

7 Maximum Principle for heat equation and Classification of second order PDEs

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. Suppose that $u \in C([0,T] \times \overline{\Omega}) \cap C^{1,2}((0,T) \times \Omega)$ with $u_t = \Delta_x u$ in $(0,T) \times \Omega$. Then u attains its maximum either on $\{0\} \times \Omega$ or on $[0,T] \times \partial\Omega$.*

Corollary 7.2. *Let $f \in C(\{0\} \times \Omega)$, $g \in C([0,T] \times \partial\Omega)$. There is at most one function $u \in C([0,T] \times \overline{\Omega}) \cap C^{1,2}((0,T) \times \Omega)$ with $u_t = \Delta_x u$ in $(0,T) \times \Omega$, $u = f$ on $\{0\} \times \Omega$ and $u = g$ on $[0,T] \times \partial\Omega$.*

Let $\Omega \subset R^n$ be open and bounded. For $u : \Omega \to \mathbb{R}$ with $u \in C^2(\Omega)$, we consider a differential operator L having the form

$$Lu = \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum b^i(x) u_{x_i} + c(x) u, \tag{21}$$

for some continuous functions $a^{ij}, b^i, c : \Omega \to \mathbb{R}$. Lu is called a partial differential operator of second order. The partial differential equation

$$Lu(x) = f(x), \quad x \in \Omega, \tag{22}$$

where $f : \Omega \to \mathbb{R}$ continuous, is called a partial differential equation of second order. If $u \in C^2(\Omega)$ then

$$\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} = Tr(A(x) D^2 u(x)),$$

where

$$D^2 u(x) := \begin{pmatrix} u_{x_1x_1}(x) & \cdots & u_{x_1x_n}(x) \\ u_{x_2x_1}(x) & \cdots & u_{x_2x_n}(x) \\ \vdots & & \vdots \\ u_{x_nx_1}(x) & \cdots & u_{x_nx_n}(x) \end{pmatrix} = \Big(\frac{\partial^2 u}{\partial x_j \partial x_k}(x)\Big)_{j,k=1}^{n}$$

is the Hessian Matrix of u at x and

$$A(x) := \Big(\frac{a^{ij}(x) + a^{ji}(x)}{2}\Big)_{j,k=1}^{n}.$$

Definition 7.3. *The partial differential equation* (22) *is called:*
(1) Elliptic in $x \in \Omega$ if all the eigenvalues of $A(x)$ have the same sign.
(2) Parabolic in $x \in \Omega$ if $A(x)$ has the eigenvalue 0.
(3) Hyperbolic in $x \in \Omega$ if $A(x)$ has $n-1$ eigenvalues of the same sign and 1 eigenvalue of the opposite sign. The partial differential equation (22) *is called elliptic/ parabolic/hyperbolic if it is elliptic/ parabolic/hyperbolic in every $x \in \Omega$.*

8 An introduction to weak derivatives and Sobolev spaces

Some notation
(i) In the entire lecture we denote by U an open subset of $\mathbb{R}^n$. $A \subset\subset U$ means that there exists a compact set K with $A \subset K \subset U$.
(ii) $C_c^\infty(U) := \{\phi : U \to \mathbb{R} \mid \phi \in C^\infty(U), \text{ and } \operatorname{supp}\phi \subset\subset U\}$, where $\operatorname{supp}\phi := \overline{\{x \in U : \phi(x) \neq 0\}}$. An element of $C_c^\infty(U)$ is called a test function.
(iii) For a multi-index is $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ we define $\alpha! = \prod_{j=1}^{n} \alpha_j!$.
(v) $L^p_{\text{loc}}(U) := \{\phi : U \to \mathbb{R} \mid \phi \in L^p(K) \text{ for all } K \subset\subset U\}$.

Definition 8.1 (**Weak derivatives**)**.** *Suppose that $u, w \in L^1_{loc}(U)$. We say that w is the α^{th}-weak partial derivative of u, and write $\partial^\alpha u = v$ if*

$$\int_U u \partial^\alpha \phi dx = (-1)^{|\alpha|} \int_U w\phi dx, \forall \phi \in C_c^\infty(U).$$

Proposition 8.2 (Uniqueness of weak derivatives)**.** *If $u \in L^1_{loc}(U)$ has a weak derivative then it is unique.*

Definition 8.3 (**Sobolev spaces**)**.** *Let $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. Then*

$$W^{k,p}(U) := \{u \in L^p(U) \mid \partial^\alpha u \text{ exists and } \partial^\alpha u \in L^p, \text{ for all multiindices } \alpha, \text{ with } |\alpha| \leqslant k\}.$$

We equip $W^{k,p}(U)$ with the norm $\|.\|_{W^{k,p}(U)}$ defined by

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \Big(\sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^p_{L^p}\Big)^{\frac{1}{p}}, & \text{if } p < \infty \\ \max_{|\alpha| \leqslant k} \|\partial^\alpha u\|_{L^\infty}, & \text{if } p = \infty. \end{cases}$$

Elementary properties of Sobolev spaces

Theorem 8.4 (Completeness). *For any $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$ the space $(W^{k,p}(U), \|.\|_{W^{k,p}(U)})$ is a Banach space.*

Theorem 8.5 (Leibnitz rule). *If $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, $u \in W^{k,p}(U)$ and $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and for all $|\alpha| \leqslant k$ we have*

$$\partial^\alpha(\zeta u) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta \zeta \partial^{\alpha-\beta} u,$$

where $\beta \leqslant \alpha$ means $\beta_j \leqslant \alpha_j$ for all $j \in \{1, \ldots, n\}$, and $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Lemma 8.6. *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in C^\infty(\mathbb{R})$ be such that $0 \leqslant f \leqslant 1$, $f(x) = 1$ if $x \leqslant 1$ and $f(x) = 0$ if $x \geqslant 2$. Define $\chi_R : \mathbb{R}^n \to \mathbb{R}$ with $\chi_R(y) = f(|y| - R)$. If $u \in W^{k,p}(\mathbb{R}^n)$ then $\chi_R u \in W^{k,p}(\mathbb{R}^n)$ for all $R > 0$ and $\lim_{R\to\infty} \chi_R u = u$ in $W^{k,p}(\mathbb{R}^n)$.*

Lemma 8.7. *1) Let $u \in L^1_{loc}(\mathbb{R}^n)$. Then $\eta_\epsilon * u \in C^\infty$. Here $\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$ with $\eta(x) := \begin{cases} ke^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geqslant 1. \end{cases}$,*
where $k > 0$ is chosen such that $\int_{R^n} \eta(x)dx = 1$.

2) If $u \in L^p(\mathbb{R}^n)$, where $1 \leqslant p < \infty$, then $\eta_\epsilon * u \in L^p(\mathbb{R}^n)$. Moreover $\|\eta_\epsilon * u\|_{L^p} \leqslant \|u\|_{L^p}$ and $\lim_{\epsilon\to 0+} \|\eta_\epsilon * u - u\|_{L^p} = 0$.

Theorem 8.8 (Approximation by smooth functions). *Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. If $u \in W^{k,p}(\mathbb{R}^n)$, then there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{k,p}(\mathbb{R}^n)$.*

9 Some tools of functional analysis

Here we will introduce some tools of the functional analysis that we will need to handle the problem of the drum in the next section. For simplicity we will explain some of the needed tools only for Hilbert spaces.

Definition 9.1. *Let $(X, \|.\|)$ be a real (respectively complex) Banach space, $x \in X$ and x_n be a sequence in x. We say that x_n converges weakly to x and we write $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ for all $f : X \to \mathbb{R}$ (respectively $f : X \to \mathbb{C}$) linear continuous.*

Remark 1. *If $(X, \langle ., . \rangle)$ is a Hilbert space then $x_n \rightharpoonup x$ if and only if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in X$.*

Theorem 9.2. *Let $(X, \langle ., . \rangle)$ be a Banach space. Then every weakly convergent sequence in X is bounded. Moreover, if $x_n \rightharpoonup x$ then $\|x\| \leqslant \liminf_{n\to\infty} \|x_n\|$.*

Theorem 9.3 (Banach-Alaoglu, special case). *Let $(X, \langle ., . \rangle)$ be a separable Hilbert space (this means it has a countable dense subset). Then every bounded sequence in X has a weakly convergent subsequence.*

Theorem 9.4. *Let $(X, \langle ., . \rangle)$ be a Hilbert space. If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.*

Definition 9.5 (Compact operators). *Let X, Y be Banach spaces and $K : X \to Y$ a linear operator. K is called compact, if for every bounded sequence x_n in X Kx_n has a convergent subsequence in Y.*

Theorem 9.6. *Let X, Y be separable Hilbert spaces, and $K : X \to Y$ be a linear operator. Then K is compact if and only if for any sequence x_n in X and any $x \in X$ we have $x_n \rightharpoonup x \implies Kx_n \to Kx$.*

Definition 9.7 ($W_0^{k,p}(U)$). *Let $U \subset \mathbb{R}^n$ open. We denote by $W_0^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.*

Lemma 9.8. *If $U, W \subset \mathbb{R}^n$ open with $U \subset W$ and $u \in W_0^{k,p}(U)$ then the function $\widetilde{u} : W \to \mathbb{R}$* $\widetilde{u}(x) = \begin{cases} u(x), x \in U \\ 0, x \notin U \end{cases}$ *is in $W_0^{k,p}(W)$.*

Lemma 9.9. *Assume that $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leqslant p < \infty$. Then*

$$\|\eta_\epsilon * u - u\|_{L^p(\mathbb{R}^n)} \leqslant \epsilon \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Corollary 9.10. *If $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ then $\|u_m * \eta_\epsilon - u_m\|_{L^p} \to 0$ as $\epsilon \to 0$ uniformly in m.*

Lemma 9.11. *Let $U \subset \mathbb{R}^n$ be open and bounded and $W = U + B_2(0)$. Assume that $(u_m)_{m\in\mathbb{N}}$ is bounded in $W_0^{1,p}(U)$. For any $\epsilon \in (0,1)$ consider the family of functions $f_m^\epsilon := \widetilde{u}_m * \eta_\epsilon$, where $\widetilde{u}_m$ is as in Lemma 9.8. Then f_m^ϵ is in $W_0^{1,p}(W) \cap C(\overline{W})$ and it is bounded and equicontinuous in $C(\overline{W})$.*

Theorem 9.12 (Arzela Ascoli, special case)**.** *Let $K \subset \mathbb{R}^n$ compact and $C(K) = \{f : K \to \mathbb{C}, f$ continuous $\}$ equipped with the norm $\|f\| = \max\{|f(x)| : x \in K\}$. If $F \subset C(K)$ is bounded and equicontinuous then every sequence in F has a subsequence that is convergent in $C(K)$.*

Theorem 9.13 (Special case of Rellich-Kondrachov Compactness)**.** *Assume that $U \subset \mathbb{R}^n$ is bounded and open. Then the identity map $i : W_0^{1,p}(U) \to L^p(U)$, where $1 \leqslant p < \infty$, is compact.*

10 Application to the equations $u_{tt} = \Delta u$, $u_t = \Delta u$ on bounded domains with Dirichlet boundary conditions

Let $U \subset \mathbb{R}^n$ open and bounded and let $X := \{u \in W_0^{1,2}(U) : \|u\|_{L^2(U)} = 1\}$. We consider the functional $E : X \to \mathbb{R}$, $E(u) = \int_U |\nabla u|^2 dx$. We are going to show that:

Theorem 10.1. *The functional E has always a minimizer $u_0 \in X$.*

Theorem 10.2. *If $u_0 \in X$ is a minimizer of E, then $-\Delta u_0 = E(u_0)u_0$, where the last equality is in the sense of weak derivatives.*

Lemma 10.3. *Let $u_0 \in X$ be a minimizer of E. Then $E|_{X\cap\{u_0\}^\perp}$ has a minimizer u_1 and $-\Delta u_1 = E(u_1)u_1$.*

Remark 2. *The orthogonal complement can be understood either in the L^2 sense or in the $W_0^{1,2}$ sense, namely the two complements are the same.*

Theorem 10.4. *The eigenvectors of the Laplacian in $W_0^{1,2}(U)$ form an orthonormal basis of $L^2(U)$.*

We will explain why this verifies the validity of the ansatz of separation of variables for the equation of a drum and the heat equation on U with Dirichlet boundary conditions.

Lemma 10.5. *If $U \subset V$ then the minimum of E on U is bigger or equal than the minimum of E on V.*

As we will explain this can be interpreted in the way: "A big drum has a lower sound than a small drum".